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SUPPLEMENT TO "A CLT FOR SECOND DIFFERENCE ESTIMATORS WITH AN APPLICATION TO VOLATILITY AND INTENSITY": PROOFS AND TECHNICAL ISSUES

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Throughout this supplement, we refer to the article "A CLT for second difference estimators with an application to volatility and intensity" as the main text.

APPENDIX A: NOTATION AND CONDITIONS

We start by recalling some definitions from the article "Assessment of uncertainty in high frequency data: The observed asymptotic variance" (Mykland and Zhang, 2017a).

DEFINITION A.1. (ORDERS IN PROBABILITY) For a sequence $\alpha_t^{(n)}$ of semimartingales, we say that $(\alpha_t^{(n)}) = O_p(1)$ if the sequence is tight, with respect to convergence in law relative to the Skorokhod topology on $\mathbb{D}[0,T]$, with $\mathbb{D}[0,T]$ the space of càdlàg functions on $[0,T]$ (Jacod and Shiryaev, 2003, Theorem VI.3.21, p. 350). For scalar random quantities, $O_p(\cdot)$ and $o_p(\cdot)$ are defined as usual, see e.g., Pollard (1984, Appendix A).

CONDITION A.1. Let $\alpha_t^{(n)}$ and $\beta_t^{(n)}$ be sequences (in n) of semimartingales. Each of these sequences are (separately) assumed to be $O_p(1)$.

DEFINITION A.2. (NOTATION). The symbol $\mathscr{F}$ will refer to a collection of nonrandom functions $f_\cdot^{(l,n)}$, càdlàg on $[0,T]$, with $n \in \mathbb{N}$, and $l = 1,...,2K$, satisfying

$$|f_t^{(l,n)}| \leq 1 \text{ for all } t, l, \text{ and } n.$$

Similarly, $\mathscr{G}$ will refer to a collection $g_t^{(l,n)}$ with the same size and properties. Given $\mathscr{F}$ and $\mathscr{G}$, set

$$\alpha_t^{(l,n)} = \int_0^t f_{s-}^{(l,n)}\,\mathrm{d}\alpha_s^{(n)} \quad \text{and} \quad \beta_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)}\,\mathrm{d}\beta_s^{(n)} \quad \text{for } l = 1,\dots,2K,$$

where $\alpha^{(n)}$ and $\beta^{(n)}$ are sequences of semimartingales satisfying Condition A.1. For two semimartingales λ and θ, their quadratic covariation is denoted $[\theta,\lambda]$, and for two locally square integrable martingales θ^{mg} and λ^{mg}, $\langle\theta^{\mathrm{mg}},\lambda^{\mathrm{mg}}\rangle$ denotes their predictable quadratic covariation, see Jacod and Shiryaev (2003, Theorem I.4.2, p. 38 and Definition I.4.45, p. 51). For a random variable $X \in L^p(\Omega,\mathcal{F},P)$ the norm is $\|X\|_p = (\mathrm{E}\,|X|^p)^{1/p}$. If $f(s)$ and $g(s)$ are defined on $[0,T]$ and $f(s) \leq cg(s)$ for all $0 \leq s \leq T$, for a fixed constant c, we write $f(s) \lesssim g(s)$. To indicate that a sequence of random variables or processes converges in law (or in distribution) to X, we write $X_n \Rightarrow X$. Stable convergence in law, defined in Definition 2 on p. 5 of the main text, is denoted $X_n \Rightarrow X$ stably.

CONDITION A.2. (CONDITIONS FOR RATE-OF-CONVERGENCE AND CLT) The sequence of semimartingales $\alpha_t^{(n)}$, possibly defined on a sequence of filtrations $(\mathcal{F}_t^n)_{0\le t\le T}$, is said to satisfy this condition if it can be decomposed as $\alpha_t^{(n)} = \alpha_0^{(n)} + \int_0^t \mu_s^{(n)} \mathrm{d}s + \bar{\alpha}_t^{(n)}$, where for each n, $\bar{\alpha}_t^{(n)}$ is a locally square integrable martingale with predictable quadratic variation $\langle \bar{\alpha}^{(n)}, \bar{\alpha}^{(n)} \rangle_t$ that is absolutely continuous, while $\mu_t^{(n)}$ and $\mathrm{d}\langle \bar{\alpha}^{(n)}, \bar{\alpha}^{(n)} \rangle_t/\mathrm{d}t$ are locally bounded uniformly in n. The process $\mu_t^{(n)}$ itself also satisfies the preceding assumptions of this condition.

We emphasize that Condition A.2 is not an infinite recursion. That is, we say nothing about the coefficients of $\mu_t^{(n)}$ other than that they are locally bounded uniformly in n.

A single semimartingale α_t is said to satisfy Condition A.2 if the above is satisfied for the constant sequence $\alpha_t = \alpha_t^{(n)}$. Note also that Condition A.2 implies that each $\alpha_t^{(n)}$ is an Itô-semimartingale (see Jacod and Protter (2012, Eq. (4.4.1), p. 114)).

REMARK A.1. Let us untangle Condition A.2 a bit. Let $\{a_n(t)\}_{n\ge 1, t\in[0,T]}$ be a sequence of adapted processes. For each $n \ge 1$, that $a_n(t)$ is *locally bounded* means that there are positive constants, say $(a_{n,m}^+)_{m\ge 1}$, and a sequence of stopping times $(\tau_{n,m})_{m\ge 1}$ so that $|a_{n,m}(t)| \le a_{n,m}^+$ a.s. for all $t \le \tau_{n,m}$, and that $P(\tau_{n,m} = T) \to 1$ as $m \to \infty$. That $\{a_n(t)\}_{n\ge 1, t\in[0,T]}$ is locally bounded *uniformly in* n means that for each $m \ge 1$, $\sup_{n\ge 1} a_{n,m}^+ \le a_m^+$ say. This assumption has two consequences that are used repeatedly in the following, particularly in the proofs in Appendix E and F. We collect these consequences in two lemmata.

LEMMA A.1. *Let $\{a_n(t)\}_{n\ge 1, t\in[0,T]}$ be as described in Remark A.1, and let $\{r_n(t)\}_{n\ge 1}$ be a sequence of adapted processes. Set $Y_n(t) = \int_0^t a_n(s) r_n(s)\,\mathrm{d}s$. If $\int_0^t |r_n(s)|\,\mathrm{d}s \to_p 0$ as $n \to \infty$ for all t, then $\sup_{t\le T} |Y_n(t)| \to_p 0$ as $n \to \infty$.*

PROOF. For any $\varepsilon > 0$ we have $P(\sup_{t\le T} |Y_n(t)| \ge \varepsilon) \le P(\sup_{t\le \tau_{n,m}} |Y_n(t)| \ge \varepsilon) + P(\tau_{n,m} \ne T)$. Since $\int_0^t |r_n(s)|\,\mathrm{d}s$ is continuous and adapted, it is predictable. And since $|Y_n(t)| \le a_{n,m}^+ \int_0^t |r_n(s)|\,\mathrm{d}s$ for all $t \le \tau_{n,m}$, Y_n is L-dominated by $a_{n,m}^+ \int_0^t |r_n(s)|\,\mathrm{d}s$. For any $\varepsilon, \eta > 0$, $P(\sup_{t\le \tau_{n,m}} |Y_n(t)| \ge \varepsilon) \le \eta/\varepsilon + P(a_{n,m}^+ \int_0^t |r_n(s)|\,\mathrm{d}s \ge \eta)$ by Lenglart's inequality (Jacod and Shiryaev, 2003, Lemma I.3.30, p. 35). By the uniformity in n, i.e., since $\sup_{n\ge 1} a_{n,m}^+ \le a_m^+$, and $\int_0^t |r_n(s)|\,\mathrm{d}s \to_p 0$, we can make $P(a_{n,m}^+ \int_0^t |r_n(s)|\,\mathrm{d}s \ge \eta) \le P(a_m^+ \int_0^t |r_n(s)|\,\mathrm{d}s \ge \eta)$ arbitrarily small by choosing n large. For each $n \ge 1$ we can make $P(\tau_{n,m} \ne T)$ arbitrarily small by choosing m large. □

The argument used in the proof above is the type of localisation argument we use repeatedly in the following. From the proof, it is also clear how the assumption of $\{a_n(t)\}_{n\ge 1, t\in[0,T]}$ being locally bounded *uniformly in* n comes into play. In the following we will also drop the index m from the bounding constants, for example, a_m^+ in the lemma is simply a^+.

The next lemma is 'local and stable' version of one of the Cramér–Slutsky rules (see e.g., van der Vaart (1998, Lemma 2.8(i), p. 11) for the finite dimensional version). This lemma is used in the proof of Theorem 3.2 in Appendix F.

LEMMA A.2. *Let $(X_n)_{n\ge 1}$ be a sequence of càdlàg processes with values in $\mathbb{D}[0,T]$, all defined on the same probability space $(\Omega, \mathcal{F}, P)$. Let $(\tau_{n,m})_{n\ge 1, m\ge 1}$ be an array of stopping times such that $P(\tau_{n,m} = T) \to 1$ as $m \to \infty$ for each n. Suppose that $X_n = Y_n + Z_n$ for each n, and that (i) $\sup_{t\le \tau_{n,m}} |Z_n(t)| = o_p(1)$ as $n \to \infty$; and (ii) Y_n converges $\mathcal{F}$-stably to Y. Then X_n converges $\mathcal{F}$-stably to Y.*

PROOF. We have that $\sup_{t\le T}|Z_n(t)|\to_p 0$ because of the inequality $P(\sup_{t\le T}|Z_n(t)|\ge \varepsilon)\le P(\sup_{t\le \tau_{n,m}}|Z_n(t)|\ge\varepsilon)+P(\tau_{n,m}\ne T)$ for any $\varepsilon>0$. Since Y_n converges stably to Y, Y_n also converges in law to Y, so by Jacod and Shiryaev (2003, Lemma VI.3.31, p. 352), X_n converges in law to Y, and so X_n is tight (Billingsley, 1999, Theorem 5.2, p. 60). Let $\{t_1,\ldots,t_k\}$ be any subset of $[0,T]$. From the $\mathcal{F}$-stable convergence of Y_n to Y, $(U,Y_n(t_1),\ldots,Y_n(t_k))\to_d (U,Y(t_1),\ldots,Y(t_k))$ for any bounded $\mathcal{F}$-measurable random variable U (Jacod and Shiryaev, 2003, Prop. VII.5.33, p. 513). Because $Z_n(t)\to_p 0$ for all t, $(U,X_n(t_1),\ldots,X_n(t_k))=(U,Y_n(t_1),\ldots,Y_n(t_k))-(0,Z_n(t_1),\ldots,Z_n(t_k))\to_d (U,Y(t_1),\ldots,Y(t_k))$ by the Cramér–Slutsky rules (van der Vaart, 1998, Lemma 2.8(i), p. 11). Combined with the tightness of (U,X_n) (special case of Corollary VI.3.33 in Jacod and Shiryaev (2003, p. 353)), this yields that $\mathrm{E}\,g(U)f(X_n)\to\mathrm{E}\,g(U)f(Y)$ for all bounded and continuous functions f and g. This means that Definition 2 on p. 5 of the main text (stable convergence) holds for all $g(U)$ with g bounded and continuous. What we need is that $\mathrm{E}\,g(U)f(X_n)\to\mathrm{E}\,g(U)f(Y)$ holds for all bounded g. Proceed as in the proof of Lemma 2.1 in Jacod and Protter (1998, p. 270). Assume that g is bounded (so not necessarily continuous), and let $(g_k)_{k\ge1}$ be a sequence of bounded and continuous functions such that $\mathrm{E}\,|g_k(U)-g(U)|\to 0$ as $k\to\infty$. Then $|\mathrm{E}\,g(U)f(X_n)-\mathrm{E}\,g(U)f(Y)|\le |\mathrm{E}\,g_k(U)f(X_n)-\mathrm{E}\,g_k(U)f(Y)|+2\sup_x|f(x)|\,\mathrm{E}\,|g(U)-g_k(U)|$ which can be made arbitrarily small by choosing n and k sufficiently large.

□

DEFINITION A.3. A sequence of processes $(\xi_t^{(n)})_{n\ge1}$ is locally continuous in mean square if for each n

$$\sup_{0\le|t-s|\le\delta}\mathrm{E}\,(\xi_t^{(n)}-\xi_s^{(n)})^2\to 0,\quad \text{as}\quad \delta\to 0,$$

for $t\vee s\le\tau_{n,m}$, where $(\tau_{n,m})_{n\ge1,m\ge1}$ are stopping times such that $P(\tau_{n,m}=T)\to 1$ as $m\to\infty$.

For the proof of Theorem 3.2 of the main text, given in Section F, we need to be more specific about the construction of the probability space on which the sequence of processes $\alpha^{(n)}$, $\beta^{(n)}$, as well as potentially stochastic spot-processes related to these two, are defined. Since the result of said theorem is a *stable* convergence result, we need everything (except, possibly, microstructure noise) to be defined on the same probability space. Let $(\Omega,\mathcal{F},\mathbb{F},P)$, with $\mathbb{F}=(\mathcal{F}_t)_{0\le t\le T}$ be a filtered probability space on which the processes are defined, and for each n let $\mathbb{F}^n=(\mathcal{F}_t^n)_{0\le t\le T}$ be a filtration on $(\Omega,\mathcal{F})$.

CONDITION A.3. A filtration $(\mathcal{F}_t)_{0\le t\le T}$ on $(\Omega,\mathcal{F})$ is said to satisfy the current condition if it is generated by $(\mu,W^{(1)},W^{(2)},\ldots)$ where μ is a Poisson random measure with deterministic compensator ν that is absolutely continuous as a function of time, and $W^{(1)},W^{(2)},\ldots$ are independent one-dimensional Wiener processes.

CONDITION A.4. For any finite family of $\mathcal{F}_t$-adapted bounded martingales $(X_1,\ldots,X_p)$ there is a sequence of $\mathcal{F}_t^n$-adapted martingales $(X_1^n,\ldots,X_p^n)$ such that $(X_1^n,\ldots,X_p^n)\to_p (X_1,\ldots,X_p)$.

By Cohen and Elliott (2015, Theorem 14.5.7, p. 360), Condition A.3 is sufficient to represent the local martingales encountered in Theorem 3.2 of the main text. Importantly, any martingale X (resp. X^n) adapted to $\mathbb{F}$ (resp. $\mathbb{F}^n$) has a predictable quadratic variation process $\langle X,X\rangle$ (resp. $\langle X^n,X^n\rangle$) that is absolutely continuous with respect to Lebesgue measure.

APPENDIX B: A STABLE CENTRAL LIMIT THEOREM FOR CÀDLÀG MARTINGALES

We find the following theorem and its corollaries to be convenient in applications. It is a generalisation of Theorem 2.28 in Mykland and Zhang (2012, p. 152) (originally stated in Zhang (2001)), and it is a special case of a theorem found in Jacod and Shiryaev (2003, Theorem IX.7.3, p. 584), but with a different and perhaps more accessible statement and proof. The proof of the present theorem employs techniques from the proofs of both these earlier theorems. The formulation of our theorem also gives rise to Corollary B.2. This corollary provides alternative Lindeberg type conditions that might be easier to check.

We have a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$. For each n, we have a filtration $\mathbb{F}^n = (\mathcal{F}_t^n)_{0\le t\le T}$ and a $\mathcal{F}_t^n$-adapted square integrable martingale $M^n = \{M_t^n \colon 0 \le t \le T\}$. This is the martingale that we wish to show that converges stably in law.

If the σ-algebra $\mathcal{F}$ is countably generated, that is, $\mathcal{F} = \sigma(A_1, A_2, \ldots)$ for a countable sequence $A_1, A_2, \ldots$ in Ω, then there is a sequence $(Y_m)_{m\ge 1}$ of random variables that is dense in $L^1(\Omega, \mathcal{F}, P)$ (Kolmogorov and Fomin, 1970, Theorem 3, p. 382). For $m = 1, 2, \ldots$ set $N_t^m = \mathrm{E}(Y_m \mid \mathcal{F}_t)$. Then each N_t^m is a bounded martingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and we have the following result, stated in Jacod (1997, p. 239),

(b) If $(\mathcal{G}_t)_{0\le t\le T}$ is the smallest filtration with respect to which $(N_t^m)_{m\ge 1}$ is adapted, then $\mathcal{G}_t = \mathcal{F}_t$ up to P-null sets.

The countably many $\mathcal{F}_t$-adapted bounded martingales N_t^m play a role similar to the Wiener processes appearing in Condition 2.26 in Mykland and Zhang (2012, p. 151).

THEOREM B.1. *Assume Condition A.4. Let $M^n = \{M_t^n \colon 0 \le t \le T\}$ be a sequence of square integrable local martingales on $(\Omega, \mathcal{F}, P)$, adapted to $\mathcal{F}_t^n$ for each n. Suppose that there is an $\mathcal{F}_t$-adapted process f_t such that*

(i) $\langle M^n, M^n\rangle_t \to_p \int_0^t f_s^2 \,\mathrm{d}s$ *for all* t*;*
(ii) $\int_{|x|>\varepsilon} x^2 \nu^n([0,T] \times \mathrm{d}x) \to_p 0$ *for all* $\varepsilon > 0$*;*
(iii) $\langle M^n, X^n\rangle_t \to_p 0$ *for all* t *and all bounded martingales* X *on* $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

Then M^n converges stably in distribution to $M_t = \int_0^t f_s \,\mathrm{d}W_s$, where W_s is a Wiener process defined on an extension of the original probability space.

PROOF. Convergence in probability implies convergence in distribution, so (i) implies that $\langle M^n, M^n\rangle_t \to_d \int_0^t f_s^2 \,\mathrm{d}s$ in the sense of finite dimensional distributions. Combining this with the facts that $\langle M^n, M^n\rangle_t$ is a non-decreasing process and has a non-decreasing and continuous limit, Theorem VI.3.37 in Jacod and Shiryaev (2003, p. 354) yields process convergence of $\langle M^n, M^n\rangle$ to $\int_0^\cdot f_s^2 \,\mathrm{d}s$. The sample paths $t \mapsto \int_0^t f_s^2 \,\mathrm{d}s$ are continuous, so $\langle M^n, M^n\rangle$ is C-tight (Jacod and Shiryaev, 2003, Def. 3.25, p. 351), which implies that M^n is tight (Jacod and Shiryaev, 2003, Theorem VI.4.12, p. 358). Condition (ii) implies that $\sup_{s\le T} |\Delta M_t^n| \to_p 0$, which, combined with the tightness of M^n, implies that M^n is C-tight (Jacod and Shiryaev, 2003, Lemma VI.4.22, p. 360, and Theorem VI.3.26(iii), p. 351). Assume now that $\mathcal{F}$ is countably generated, and let $(Y_m)_{m\ge 1}$ be dense in $L^1(\Omega, \mathcal{F}, P)$. Set $N_t^m = \mathrm{E}(Y^m \mid \mathcal{F}_t)$, and denote $\mathscr{N} = (N^m)_{m\ge 1}$. By Condition A.4 there is a sequence $\mathscr{N}^n = (N_1^n, N_2^n, \ldots)$, such that $\mathscr{N}^n \to_p \mathscr{N}$. Since M^n is C-tight and $\mathscr{N}^n = (N_1^n, N_2^n, \ldots)$ is tight by Condition A.4, Corollary 3.33 in Jacod and Shiryaev (2003, p. 353) gives that $(M^n, \mathscr{N}^n)$ is tight. By Prokhorov's theorem (see e.g., (Billingsley, 1999, Theorem 5.1, p. 59)), this tightness entails that we can for any subsequence n_k find a further subsequence n_{k_j} such that

$$(M^{n_{k_j}}, \mathscr{N}^{n_{k_j}}) \Rightarrow (M, \mathscr{N}). \tag{B.1}$$

For each n, write

$$M_t^n = M_t^{n,b} + xI\{|x|>1\} \star (\mu^n - \nu^n)_t, \tag{B.2}$$

in terms of the measure μ^n associated with the jumps of M^n, and its compensator ν^n, and where $M_t^{n,b}$ is a local martingale with bounded jumps. For the decomposition in (B.2), see e.g., Jacod and Protter (2012, Eq. (2.1.10), p. 29) and use that M^n, their X, is a martingale; or see Proposition II.2.29 in Jacod and Shiryaev (2003, p. 82), and the fact that their $A \equiv 0$ in the martingale case. Since $xI\{|x|>1\} \star \nu^n$ is the predictable compensator of $xI\{|x|>1\} \star \mu^n$, it follows from Lenglart's inequality (Jacod and Shiryaev, 2003, Lemma 3.30(a), p. 35) and Condition (ii) that $xI\{|x|>1\} \star \mu_t^n \to_p 0$ for all $t \in [0,T]$, thus

$$\sup_{t \le T} |M_t^n - M_t^{n,b}| \xrightarrow{p} 0. \tag{B.3}$$

But (B.3) must also hold for any subsequence, so (B.1) and the Cramér–Slutsky rules entail that $(M^{n_{k_j},b}, \mathscr{N}^{n_{k_j}})$ converges in law to $(M, \mathscr{N})$. Since $M^{n,b}$ has bounded jumps $|\Delta M^{n,b}| \le 1$, Theorem IX.1.17 in Jacod and Shiryaev (2003, p. 526) gives that M is a local martingale with respect to the filtration generated by $\mathscr{N}$ (hence the importance of fact (b), and where we use that Theorem IX.1.17 extends from the finite to the countable case, see Jacod and Shiryaev (2003, p. 586)).

We now want to show that M^n is P-UT, because that will ensure joint convergence of $(M^{n_{k_l}}, [M^{n_{k_l}}, M^{n_{k_l}}])$. Let $H^n \in \mathscr{H}^n$, where $\mathscr{H}^n$ as well as the elementary stochastic integral $H^n \cdot M_t^n$ are as defined in Jacod and Shiryaev (2003, p. 377). Then $\mathrm{E}\,|H^n \cdot M_t^n|^2 = \mathrm{E}\,(H^n)^2 \cdot [M^n, M^n]_t \le \mathrm{E}\,[M^n, M^n]_t = \mathrm{E}\,\langle M^n, M^n \rangle_t$. So by Lenglart's inequality Jacod and Shiryaev (2003, Lemma I.3.30(a), p. 35), for every t, and for any $H^n \in \mathscr{H}$, and for any $a, \eta > 0$,

$$P(|H^n \cdot M_t^n| \ge a) \le P(\sup_{0 \le t \le T} |H^n \cdot M_t^n| \ge a) \le \frac{\eta}{a^2} + P(\langle M^n, M^n \rangle_t \ge \eta).$$

But since $\langle M^n, M^n \rangle_t$ is tight, this shows that M^n is P-UT. Since M^n is P-UT, Theorem VI.6.26 in Jacod and Shiryaev (2003, p. 384) gives that $(M^{n_{k_l}}, [M^{n_{k_l}}, M^{n_{k_l}}])$ converges in law to $(M, [M, M])$; from continuity of M we get that $[M, M] = \langle M, M \rangle$ (Jacod and Shiryaev, 2003, Theorem I.4.52, p. 55); and by Condition (i), $\langle M, M \rangle_t = \int_0^t f_s^2 \,\mathrm{d}s$.

Assume without loss of generality that $f_s > 0$ (see Mykland and Zhang (2012, p. 152)), and set $W_t = \int_0^t f_s^{-1/2} \,\mathrm{d}M_s$. Then $\langle W, W \rangle_t = t$ and by Condition (iii) the quadratic covariation $\langle W, X \rangle_t = \int_0^t f_s^{-1/2} \langle M, X \rangle_t = 0$ for any bounded martingale X. Lévy's theorem (Jacod and Shiryaev, 2003, p. 102) then gives that W is a Wiener process. Since W is independent of $\mathcal{F}$ by Condition (iii), we can realise W on the extension $\widetilde{\Omega} = \Omega \times C[0,T]$, $\widetilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}$, $\widetilde{\mathcal{F}}_t = \cap_{s>t} \mathcal{F}_s \otimes \mathcal{B}_s$, $\widetilde{P}(\omega, \mathrm{d}x) = P(\mathrm{d}\omega) Q(\omega, x)$. Here $C[0,T]$ is the space of all continuous functions on $[0,T]$, $\mathcal{B}$ is the Borel-σ-algebra and $(\mathcal{B}_t)_{0 \le t \le T}$ the filtration, and for ω fixed, $Q(\omega, \mathrm{d}x)$ is the Wiener measure on $(C[0,T], \mathcal{B})$ (see Billingsley (1999, Ch. 2)). Then, for each ω, $W(\omega, x) = W(\omega)$ is a Wiener process relative to $(\mathcal{B}_t)_{0 \le t \le T}$, and $M_t(\omega) = \int_0^t f_s(\omega) \,\mathrm{d}W_s(\omega)$ is a continuous process on the extension, orthogonal to all bounded martingales on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, and $\langle M, M \rangle_t = \int_0^t f_s^2 \,\mathrm{d}s$ is $\mathcal{F}$-measurable by Condition (i). Thus, M is an $\mathcal{F}$-conditional Gaussian martingale on the extension. This proves the theorem for a subsequence n_{k_j}, but since the subsequence was arbitrary, the claim of the theorem follows (see the corollary on p. 337 in Billingsley (1995), or Billingsley (1999, Theorem 2.6, p. 20)). Finally, the assumption of $\mathcal{F}$ being countably generated can be removed using the techniques in Jacod and Shiryaev (2003, Step (5), p. 588). □

COROLLARY B.2. *Assume* (i) *and* (iii) *of Theorem B.1. If Condition* (ii) *of that theorem is replaced by one of the following conditions,*

(ii)′ $\mathrm{E}\sum_{s\le t}|\Delta M_s^n|^2 I\{|\Delta M_s^n|\ge\varepsilon\}\to 0$ *for all* $\varepsilon>0$ *and for all* t*;*
(ii)″ $\sup_{t\le T}|\Delta M_t^n|\to_p 0$*, and* $\mathrm{E}\sup_{t\le T}|\Delta M_t^n|^2<\infty$ *for all* n*;*

the conclusion of Theorem B.1 still holds.

PROOF. For (ii)′: By Proposition II.1.28 (p. 72) and Theorem I.3.17 (p. 32) in Jacod and Shiryaev (2003), we have that

$$\mathrm{E}\sum_{s\le t}|\Delta M_s^n|^2 I\{|\Delta M_s^n|\ge\varepsilon\}=\mathrm{E}\int_{|x|\ge\varepsilon}|x|^2\mu^n([0,t]\times\mathrm{d}x)=\mathrm{E}\int_{|x|\ge\varepsilon}|x|^2\nu^n([0,t]\times\mathrm{d}x),$$

which proves that (ii)′⇒(ii). For (ii)″: We must show that (ii)″ implies (B.3). Using the triangle inequality and the fact that ν_t^n is a measure for all t

$$\begin{aligned}|M_t^n-M_t^{n,b}|&\le\sum_{s\le t}|\Delta M_s^n|I\{|\Delta M_s^n|>1\}+|x|I\{|x|>1\}\star\nu_t^n\\&\le\sup_{s\le t}|\Delta M_s^n|\sum_{s\le t}I\{|\Delta M_s^n|>1\}+|x|I\{|x|>1\}\star\nu_t^n.\end{aligned}\tag{B.4}$$

Since $\langle M^n,M^n\rangle_t$ is tight, M^n is P-UT, and we have that $\sum_{s\le t}I\{|\Delta M_s^n|>1\}=O_p(1)$ for all $t>0$ (Jacod and Shiryaev, 2003, Theorem VI.6.16, p. 380), so the first term on the right in (B.4) tends to zero in probability by the Cramér–Slutsky rules. For the second term on the right, since $|x|I\{|x|>1\}\star\nu_t^n$ is the predicable compensator of the adapted process $\sum_{s\le t}|\Delta M_s^n|I\{|\Delta M_s^n|>1\}=|x|I\{|x|>1\}\star\mu_t^n$, Lenglart's inequality (Jacod and Shiryaev, 2003, Lemma I.3.30(b), p. 35) gives that for all $\varepsilon,\eta>0$,

$$P(|x|I\{|x|>1\}\star\nu_t^n\ge\varepsilon)\le\frac{1}{\varepsilon}\big(\eta+\mathrm{E}\sup_{s\le t}|\Delta M_s^n|\big)+P(|x|I\{|x|>1\}\star\mu_t^n\ge\eta).$$

As we saw above $P(|x|I\{|x|>1\}\star\mu_t^n\ge\eta)\to 0$. For all $\sigma>0$, by Hölder's inequality

$$\begin{aligned}\mathrm{E}\sup_{s\le t}|\Delta M_s^n|&\le\mathrm{E}\sup_{s\le t}|\Delta M_s^n|I\{|\Delta M_s^n|\ge\sigma\}+\sigma\\&\le(\mathrm{E}\sup_{s\le t}|\Delta M_s^n|^2)^{1/2}P(\sup_{s\le t}|\Delta M_s^n|\ge\sigma)^{1/2}+\sigma\to\sigma,\end{aligned}$$

as $n\to\infty$. Since ε,η,σ were arbitrary, $|x|I\{|x|>1\}\star\nu_t^n$ converges in probability to zero for all $t>0$. □

REMARK B.1. In the proof of Theorem B.1 we used that if M^n is a sequence of square integrable local martingales, then $\langle M^n,M^n\rangle_t\to_p\langle M,M\rangle_t$ for all t as $n\to\infty$ with $\langle M,M\rangle_t$ continuous, entails that M^n is P-UT. This is a useful implication that, perhaps because it is deemed obvious, is not spelled out explicitly in Jacod and Shiryaev (2003, ch. VI.6). Using this implication, an immediate corollary to Proposition 6 in Mykland and Zhang (2017b, p. 12) is: If M^n converges in law to M, and $\langle M^n,M^n\rangle_t\to_p V$, with V being continuous, then M^n converges $\mathcal{G}$-stably in law, where $\mathcal{G}=\sigma(V)$. For several other results associated with the P-UT property, see Mykland and Zhang (2017b, Appendix D).

APPENDIX C: PROOF OF THE CLAIMS IN EXAMPLE 1 OF THE MAIN TEXT

Assume that $\xi^n = n\xi$, $\nu_n = \sqrt{n}\nu$ and that $0 < \beta \le \beta_n \to \infty$ as $n \to \infty$. Then, for each $t \in [0,T]$,

$$\frac{1}{n}\Lambda_{n,t} \xrightarrow{p} \xi t, \quad \text{and} \quad \frac{1}{n}[\sigma^2, \lambda_n]_t \xrightarrow{p} \rho\gamma\nu\xi^{1/2}\int_0^t \sigma_s \,\mathrm{d}s, \tag{C.1}$$

as $n \to \infty$. We now prove (C.1): The expectation of the intensity is $\mathrm{E}\,\lambda_{n,t} = \xi_n$, and

$$\begin{aligned} \mathrm{E}\left|\frac{1}{n}(\lambda_{t,n} - \xi_n)\right|^2 &= \frac{1}{n^2}\,\mathrm{E}\left|\nu_n \int_0^t \lambda_{n,s}^{1/2} e^{-\beta_n(t-s)}\,\mathrm{d}B_s\right|^2 = \frac{\nu_n^2}{n^2}\,\mathrm{E}\int_0^t \lambda_{n,s} e^{-2\beta_n(t-s)}\,\mathrm{d}s \\ &= \frac{\nu_n^2}{n^2}\int_0^t \xi_n e^{-2\beta_n(t-s)}\,\mathrm{d}s = \frac{\nu_n^2}{n^2}\frac{\xi_n}{2\beta_n}\left(1 - e^{-2\beta_n t}\right). \end{aligned} \tag{C.2}$$

Note that $\mathrm{E}\,|\lambda_{t,n}/n|^2 = \mathrm{E}\,|\lambda_{t,n}/n - \xi + \xi|^2 = \mathrm{E}\,|\lambda_{t,n}/n - \xi|^2 + 2\,\mathrm{E}\,|\lambda_{t,n}/n - \xi|\xi + \xi^2 \le \mathrm{E}\,|\lambda_{t,n}/n - \xi|^2 + 2\left(\mathrm{E}\,|\lambda_{t,n}/n - \xi|^2\right)^{1/2}\xi + \xi^2$, and from (C.2), $\mathrm{E}\,|\lambda_{t,n}/n - \xi|^2 = \nu^2(2\beta_n)^{-1}\left(1 - e^{-2\beta_n t}\right)$, from which it follows that for each t, $\sup_n \mathrm{E}\,|\lambda_{t,n}/n|^2 < \infty$. By Chebyshev's inequality we get that for each t the sequence of random variables $\{\lambda_{t,n}/n\}_{m\ge1}$ are uniformly integrable (see Eq. (25.13) in Billingsley (1995, p. 338)). Moreover, from the above we see that $\mathrm{E}\,|\lambda_{t,n}/n| \le (\nu^2/\beta + \xi^2)^{1/2}$ for all t and n, and the right hand side is trivially integrable on $[0,T]$. Hence, the sequence of stochastic processes $\{\lambda_{n,s}/n\}_{n\ge1}$ satisfies the conditions of Andersen et al. (1993, Proposition II.5.2, p. 85), and the first part of (C.1) follows. For the second part we have that $\mathrm{E}\,|\sigma_t\lambda_{n,t}^{1/2}/\sqrt{n}|^2 = \mathrm{E}\,\sigma_t^2\lambda_{n,t}/n = \mathrm{E}\,|(\sigma_t^2 - \alpha + \alpha)||(\lambda_{n,t}/n - \xi + \xi)| = \mathrm{E}\,|(\sigma_t^2 - \alpha)(\lambda_{n,t}/n - \xi)| + \mathrm{E}\,|(\sigma_t^2 - \alpha)|\xi + \mathrm{E}\,|(\lambda_{n,t}/n - \xi)|\alpha + \alpha\xi$, which by three applications of Hölder's inequality and the Itô isometry is seen to be bounded by a constant, hence $\sup_n \mathrm{E}\,|\sigma_t\lambda_{n,t}^{1/2}/\sqrt{n}|^2 < \infty$, and uniform integrability of the random variables $\sigma_t\lambda_{n,t}^{1/2}/\sqrt{n}$ follows. Since $\mathrm{E}\,|\sigma_s\lambda_{n,s}^{1/2}/\sqrt{n}| \le (\mathrm{E}\,|\sigma_s\lambda_{n,s}^{1/2}/\sqrt{n}|^2)^{1/2}$ for all s and n, and a constant is integrable on $[0,T]$, so the second part of (C.1) follows by the same argument as above.

APPENDIX D: NOTES ON THEOREM 2.1 OF THE MAIN TEXT

The proof follows with trivial adjustments from Mykland and Zhang (2017a, Theorem 3, p. 208). Note that the convergence rates change due to Theorem 3.1 of the main text. We also recall the setup in Eq. (2.4) of the main text, that is $\widehat{\Theta}^n_{(s,t]} - \Theta_{(s,t]} = M^\theta_{n,t} - M^\theta_{n,s} + e^\theta_{n,t} - e^\theta_{n,s}$ and $\widehat{\Lambda}^n_{(s,t]} - \Lambda_{(s,t]} = M^\lambda_{n,t} - M^\lambda_{n,s} + e^\lambda_{n,t} - e^\lambda_{n,s}$. Mykland and Zhang (2017a, Theorem 3, p. 208) and the convergence rates from Theorem 3.1 of the main text give

$$\begin{aligned} \mathrm{QV}_{B,K}(\widehat{\Theta}^n, \widehat{\Lambda}^n) &= \overline{\mathrm{QV}}_{B,K}(\widehat{\Theta}^n, \widehat{\Lambda}^n) + R_{n,k}(\Theta, \Lambda) \\ &\quad + O_p\left((K\Delta_n + n^{-\alpha})R_{n,k}(\Lambda)^{1/2}\right) + O_p\left((K\Delta_n + n^{-\beta})R_{n,k}(\Theta)^{1/2}\right), \end{aligned} \tag{D.1}$$

where $R_{n,k}(\Theta) = R_{n,k}(\Theta, \Theta)$ and

$$R_{n,k}(\Theta, \Lambda) = \frac{1}{K}\sum_{i=K}^{B-K} (e^\theta_{n,t_{i+K}} - e^\theta_{n,t_i} - (e^\theta_{n,t_i} - e^\theta_{n,t_{i-K}}))(e^\lambda_{n,t_{i+K}} - e^\lambda_{n,t_i} - (e^\lambda_{n,t_i} - e^\lambda_{n,t_{i-K}})),$$

while $\overline{\mathrm{QV}}_{B,K}(\widehat{\Theta}^n, \widehat{\Lambda}^n)$ is given by

$$\begin{aligned}\overline{\mathrm{QV}}_{B,K}(\widehat{\Theta}^n, \widehat{\Lambda}^n) = 2[M_n^\theta, M_n^\lambda]_T + \frac{2}{3}(K\Delta_n)^2\left(1 - \frac{1}{K^2}\right)[\theta,\lambda]_T + O_p\big(n^{-(\alpha+\beta)}(K\Delta_n)^{1/2}\big)\\ + \Delta_n^2\int_0^T \left\{\left(\frac{t^*(s)-s}{\Delta_n}\right)^2 + \left(\frac{s-t_*(s)}{\Delta_n}\right)^2\right\} \mathrm{d}[\theta,\lambda]_s + O_p\big((K\Delta_n)^{5/2}\big)\\ + \Delta_n\int_0^T\left(1 - 2\frac{s-t_*(s)}{\Delta_n}\right)\mathrm{d}[\theta, M_n^\lambda]_s + O_p\big(n^{-\beta}(K\Delta_n)^{3/2}\big)\\ + \Delta_n\int_0^T\left(1 - 2\frac{s-t_*(s)}{\Delta_n}\right)\mathrm{d}[\lambda, M_n^\theta]_s + O_p\big(n^{-\alpha}(K\Delta_n)^{3/2}\big),\end{aligned}$$

where $t_*(s) = \max\{t_i : t_i < s\}$ and $t^*(s) = \min\{t_i : t_i \geq s\}$. We now consider two different sets of restrictions on the edge effect. All other cases can be deduced from (D.1). For all t on a given grid,

$$\text{Case (1): } e_{n,t}^\theta = o_p((K\Delta_n)^{1/2}n^{-\alpha}), \quad \text{and} \quad e_{n,t}^\lambda = o_p((K\Delta_n)^{1/2}n^{-\beta});$$
$$\text{Case (2): } e_{n,t}^\theta = o_p((K\Delta_n)^{3/4}n^{-\alpha}), \quad \text{and} \quad e_{n,t}^\lambda = o_p((K\Delta_n)^{3/4}n^{-\beta}).$$

Under Case (1) we have that (D.1) is

$$\mathrm{QV}_{B,K}(\widehat{\Theta}^n, \widehat{\Lambda}^n) = 2[M_n^\theta, M_n^\lambda]_T + \frac{2}{3}(K\Delta_n)^2[\theta,\lambda]_T + o_p\big((K\Delta_n)^2\big) + o_p(n^{-(\alpha+\beta)}).$$

While under Case (2) we find that (D.1) is

$$\begin{aligned}\mathrm{QV}_{B,K}(\widehat{\Theta}^n, \widehat{\Lambda}^n) = 2[M_n^\theta, M_n^\lambda]_T + \frac{2}{3}(K\Delta_n)^2[\theta,\lambda]_T\\ + O_p\big((K\Delta_n)^{5/2}\big) + O_p\big((K\Delta_n)^{1/2}n^{-(\alpha+\beta)}\big).\end{aligned}$$

It thus appears that the more stringent conditions on the edge effects in Case (2) are needed for the convergence rates of Theorem 3.1 to 'enter' Theorem 2.1 (both theorems of the main text). Do note, however, that this may be an artefact of the Cauchy–Schwarz inequality used in deriving (D.1).

APPENDIX E: PROOF OF THEOREM 3.1 OF THE MAIN TEXT

For a given K, and for $l = 1, \ldots, 2K$, set

$$t_{*,l} = t_{*,l}(s) = \max\{t_{i+K} \colon t_{i+K} \leq s, i \equiv l[2K]\}. \tag{E.1}$$

In light of the developments in the proof of Theorem 3.2 in Appendix F, in particular Eq. (F.4), it is enough to show the result when the sequences $\alpha^{(n)}$ and $\beta^{(n)}$ are local square integrable martingales. Let

$$\begin{aligned}Z_{n,l}(s) = \sum_{t_{i+K}\leq s,\, i\equiv l[2K]} (\alpha_{t_{i+K}}^{(l,n)} - \alpha_{t_{i-K}}^{(l,n)})(\beta_{t_{i+K}}^{(l,n)} - \beta_{t_{i-K}}^{(l,n)})\\ + (\alpha_s^{(l,n)} - \alpha_{t_{*,l}}^{(l,n)})(\beta_s^{(l,n)} - \beta_{t_{*,l}}^{(l,n)}) - [\alpha^{(l,n)}, \beta^{(l,n)}]_s,\end{aligned}$$

and set $Z_n(s) = (2K)^{-1}\sum_{l=1}^{2K} Z_{n,l}(s)$. Let the stopping times $(\tau_{n,m})_{n\geq 1, m\geq 1}$ be as defined in Condition A.2. That is, there are positive constants $a_{m,+}$ and $b_{m,+}$ such that, for $t \leq \tau_{n,m}$, $\mathrm{d}\langle\alpha^{(n)}, \alpha^{(n)}\rangle_t/\mathrm{d}t \leq a_{m,+}^2$ and $\mathrm{d}\langle\beta^{(n)}, \beta^{(n)}\rangle_t/\mathrm{d}t \leq b_{m,+}^2$. In particular, $|\mathrm{d}\langle\alpha^{(n)}, \beta^{(n)}\rangle_t/\mathrm{d}t| \leq$

$a_{m,+}b_{m,+}$, by the Kunita–Watanabe inequality (see e.g., Protter (2004, Theorem II.25, p. 69)). Since the index m is immaterial for what follows (see Lemma A.1), we now simply write $a_{m,+} = a_+$ and $b_{m,+} = b_+$. By Itô's lemma we have that

$$\begin{aligned}\langle Z_{n,l_1}, Z_{n,l_2}\rangle_{\tau_{n,m}} = &\int_0^{\tau_{n,m}} (\alpha_s^{(l_1,n)} - \alpha_{t_{*,l_1}}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\beta^{(l_1,n)}, \beta^{(l_2,n)}\rangle_s\\ &+ \int_0^{\tau_{n,m}} (\beta_s^{(l_1,n)} - \beta_{t_{*,l_1}}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\alpha^{(l_1,n)}, \alpha^{(l_2,n)}\rangle_s\\ &+ \int_0^{\tau_{n,m}} (\alpha_s^{(l_1,n)} - \alpha_{t_{*,l_1}}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\beta^{(l_1,n)}, \alpha^{(l_2,n)}\rangle_s\\ &+ \int_0^{\tau_{n,m}} (\beta_s^{(l_1,n)} - \beta_{t_{*,l_1}}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\alpha^{(l_1,n)}, \beta^{(l_2,n)}\rangle_s,\end{aligned}$$

from which

$$\mathrm{E}\,\langle Z_n, Z_n\rangle_{\tau_{n,m}} = \frac{1}{4K^2}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\mathrm{E}\,\langle Z_{n,l_1}, Z_{n,l_2}\rangle_{\tau_{n,m}}$$

$$\begin{aligned}= \frac{1}{4K^2}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\mathrm{E}\bigg\{&\int_0^{\tau_{n,m}} (\alpha_s^{(l_1,n)} - \alpha_{t_{*,l_1}}^{(l_1,n)})(\alpha_s^{(l_2,n)} - \alpha_{t_{*,l_1}}^{(l_2,n)})\,\mathrm{d}\langle\beta^{(l_1,n)}, \beta^{(l_2,n)}\rangle_s[2]\\ &+ \int_0^{\tau_{n,m}} (\alpha_s^{(l_1,n)} - \alpha_{t_{*,l_1}}^{(l_1,n)})(\beta_s^{(l_2,n)} - \beta_{t_{*,l_1}}^{(l_2,n)})\,\mathrm{d}\langle\beta^{(l_1,n)}, \alpha^{(l_2,n)}\rangle_s[2]\bigg\}.\end{aligned} \tag{E.2}$$

Changing the order of summation and integration we have that,

$$\begin{aligned}&\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\mathrm{E}\int_0^{\tau_{n,m}} (\alpha_s^{(l_1,n)} - \alpha_{t_{*,l_1}}^{(l_1,n)})g_s^{(l_1,n)}(\alpha_s^{(l_2,n)} - \alpha_{t_{*,l_1}}^{(l_2,n)})g_s^{(l_2,n)}\,\mathrm{d}\langle\beta^{(n)}, \beta^{(n)}\rangle_s\\ &= \int_0^T \mathrm{E}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}(\alpha_{s\wedge\tau_{n,m}}^{(l_1,n)} - \alpha_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_1,n)})g_s^{(l_1,n)}(\alpha_{s\wedge\tau_{n,m}}^{(l_2,n)} - \alpha_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_2,n)})g_s^{(l_2,n)}\,\mathrm{d}\langle\beta^{(n)}, \beta^{(n)}\rangle_s\\ &= \int_0^T \mathrm{E}\left(\sum_{l=1}^{2K}(\alpha_{s\wedge\tau_{n,m}}^{(l,n)} - \alpha_{t_{*,l}\wedge\tau_{n,m}}^{(l,n)})g_s^{(l,n)}\right)^2\mathrm{d}\langle\beta^{(n)}, \beta^{(n)}\rangle_s\\ &\le b_+^2\int_0^T \mathrm{E}\left(\sum_{l=1}^{2K}(\alpha_{s\wedge\tau_{n,m}}^{(l,n)} - \alpha_{t_{*,l}\wedge\tau_{n,m}}^{(l,n)})g_s^{(l,n)}\right)^2\mathrm{d}s,\end{aligned}$$

and similarly for the second term on the right in (E.2).

For the third and fourth terms on the right in (E.2) we use that $\mathrm{d}\langle\beta^{(n)},\alpha^{(n)}\rangle_s/\mathrm{d}s \le a_+b_+$ for $s \le \tau_{n,m}$, and Hölder's inequality,

$$
\begin{aligned}
&\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} \mathrm{E}\int_0^{\tau_{n,m}} (\alpha_s^{(l_1,n)} - \alpha_{t_{*,l_1}}^{(l_1,n)}) g_s^{(l_1,n)} (\beta_s^{(l_2,n)} - \beta_{t_{*,l_1}}^{(l_2,n)}) f_s^{(l_2,n)}\,\mathrm{d}\langle\beta^{(n)},\alpha^{(n)}\rangle_s \\
&= \int_0^T \mathrm{E}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l_1,n)} - \alpha_{t_{*,l_1},\wedge\tau_{n,m}}^{(l_1,n)}) g_s^{(l_1,n)} (\beta_{s\wedge\tau_{n,m}}^{(l_2,n)} - \beta_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_2,n)}) f_s^{(l_2,n)}\,\mathrm{d}\langle\beta^{(n)},\alpha^{(n)}\rangle_s \\
&= \int_0^T \mathrm{E}\left(\sum_{l_1=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l_1,n)} - \alpha_{t_{*,l_1},\wedge\tau_{n,m}}^{(l_1,n)}) g_s^{(l_1,n)}\right)\left(\sum_{l_2=1}^{2K} (\beta_{s\wedge\tau_{n,m}}^{(l_2,n)} - \beta_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_2,n)}) f_s^{(l_2,n)}\right)\mathrm{d}\langle\beta^{(n)},\alpha^{(n)}\rangle_s \\
&\le a_+b_+\int_0^T \mathrm{E}\left(\sum_{l_1=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l_1,n)} - \alpha_{t_{*,l_1},\wedge\tau_{n,m}}^{(l_1,n)}) g_s^{(l_1,n)}\right)\left(\sum_{l_2=1}^{2K} (\beta_{s\wedge\tau_{n,m}}^{(l_2,n)} - \beta_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_2,n)}) f_s^{(l_2,n)}\right)\mathrm{d}s \\
&\le a_+b_+\int_0^T \|\sum_{l_1=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l_1,n)} - \alpha_{t_{*,l_1},\wedge\tau_{n,m}}^{(l_1,n)}) g_s^{(l_1,n)}\|_2 \,\|\sum_{l_2=1}^{2K} (\beta_{s\wedge\tau_{n,m}}^{(l_2,n)} - \beta_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_2,n)}) f_s^{(l_2,n)}\|_2\,\mathrm{d}s,
\end{aligned}
$$

and similarly for the fourth term. Now, since $|f_s^{(l,n)}| \le 1$ and $|g_s^{(l,n)}| \le 1$ for all s, n, and l,

$$
\begin{aligned}
&\|\sum_{l=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l,n)} - \alpha_{t_{*,l}\wedge\tau_{n,m}}^{(l,n)}) g_s^{(l,n)}\|_2^2 \\
&\qquad = \mathrm{E}\sum_{l=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l,n)} - \alpha_{t_{*,l}\wedge\tau_{n,m}}^{(l,n)}) g_s^{(l,n)} \sum_{l=1}^{2K} (\alpha_{s\wedge\tau_{n,m}}^{(l,n)} - \alpha_{t_{*,l}\wedge\tau_{n,m}}^{(l,n)}) g_s^{(l,n)} \\
&\qquad = \sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} \mathrm{E}\,(\alpha_{s\wedge\tau_{n,m}}^{(l_1,n)} - \alpha_{t_{*,l_1}\wedge\tau_{n,m}}^{(l_1,n)})(\alpha_{s\wedge\tau_{n,m}}^{(l_2,n)} - \alpha_{t_{*,l_2}\wedge\tau_{n,m}}^{(l_2,n)}) g_s^{(l_1,n)} g_s^{(l_2,n)} \\
&\qquad = \sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} \mathrm{E}\,\{(\langle\alpha^{(l_1,n)},\alpha^{(l_2,n)}\rangle_{s\wedge\tau_{n,m}} - \langle\alpha^{(l_1,n)},\alpha^{(l_2,n)}\rangle_{(t_{*,l_1}\vee t_{*,l_2})\wedge\tau_{n,m}}) g_s^{(l_1,n)} g_s^{(l_2,n)}\} \\
&\qquad = \sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} \mathrm{E}\int_{(t_{*,l_1}\vee t_{*,l_2})\wedge\tau_{n,m}}^{s\wedge\tau_{n,m}} f_u^{(l_1,n)} f_u^{(l_2,n)}\,\mathrm{d}\langle\alpha^{(n)},\alpha^{(n)}\rangle_u g_u^{(l_1,n)} g_u^{(l_2,n)} \\
&\qquad \le a_+^2 \sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} (s - (t_{*,l_1} \vee t_{*,l_2})).
\end{aligned}
$$

For $l = 1, \dots, 2K$, define

$$
h_s^{(l,n)} = \sum_{K\le i\le B-K, i\equiv l[2K]} (s - t_{i-K}) I\{t_{i-K} \le s < t_{i+K}\},
$$

and notice that $(s-t_{*,l}) = (s-t_{*,l}(s)) = h_s^{(l,n)}$. Substituting the bound on $\|\sum_{l=1}^{2K}(\alpha^{(l,n)}_{s\wedge\tau_{n,m}} - \alpha^{(l,n)}_{t_{*,l}\wedge\tau_{n,m}})g_s^{(l,n)}\|_2^2$ and the three similar terms into $\mathrm{E}\,\langle Z_n, Z_n\rangle_{\tau_{n,m}}$, we get

$$
\begin{aligned}
\mathrm{E}\,\langle Z_n, Z_n\rangle_{\tau_{n,m}} &\lesssim \frac{1}{K^2}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\int_0^T (s-(t_{*,l_1}\vee t_{*,l_2}))\,\mathrm{d}s \\
&= \frac{1}{K^2}\sum_{l_1=1}^{2K}\Big\{\int_0^T (s-(t_{*,l_1}\vee t_{*,1}))\,\mathrm{d}s + \cdots + \int_0^T (s-(t_{*,l_1}\vee t_{*,2K}))\,\mathrm{d}s\Big\} \\
&\le \frac{1}{K^2}\sum_{l_1=1}^{2K}\Big\{\int_0^T (s-t_{*,l_1})\,\mathrm{d}s + \cdots + \int_0^T (s-t_{*,l_1})\,\mathrm{d}s\Big\} \\
&= \frac{2}{K}\sum_{l=1}^{2K}\int_0^T (s-t_{*,l})\,\mathrm{d}s \\
&= \frac{2}{K}\sum_{l=1}^{2K}\int_0^T h_s^{(l,n)}\,\mathrm{d}s = \frac{2}{K}\sum_{l=1}^{2K}\sum_{K\le i\le B-K,\equiv l[2K]}\int_{t_{i-K}}^{t_{i+K}} (s-t_{i-K})\,\mathrm{d}s \\
&= \frac{2}{K}\sum_{i=K}^{B-K}\int_{t_{i-K}}^{t_{i+K}} (s-t_{i-K})\,\mathrm{d}s = \frac{4}{K}\sum_{i=K}^{B-K}(K\Delta_n)^2 = 4TK\Delta_n,
\end{aligned}
$$

where the proportionality constant left out is $\max(a_+^4, b_+^4)$. By Condition A.2, $\tau_{n,m}\to_p T$ as $m\to\infty$ for each $n\ge 1$. Let $\varepsilon>0$ and choose m sufficiently large, so that $P(\tau_{n,m}\ne T)\le \varepsilon/2$, and let $c=\max(a_+^4, b_+^4)$. Then, using Markov's inequality

$$
\begin{aligned}
P(\langle Z_n, Z_n\rangle_T/(4cTK\Delta_n) > M) &\le P(\langle Z_n, Z_n\rangle_{\tau_{n,m}}/(4cTK\Delta_n) > M) + P(\tau_{n,m}\ne T) \\
&\le M^{-1}\mathrm{E}\,[\langle Z_n, Z_n\rangle_{\tau_{n,m}}/(4cTK\Delta_n)] + P(\tau_{n,m}\ne T) \\
&= M^{-1} + \varepsilon/2 \le \varepsilon,
\end{aligned}
$$

provided $M\ge 2/\varepsilon$. This shows that $\langle Z_n, Z_n\rangle_T/(4cTK\Delta_n)$ is tight, so

$$\langle Z_n, Z_n\rangle_T = O_p(4cTK\Delta_n) = O_p(K\Delta_n).$$

By Lenglart's inequality (Andersen et al., 1993, p. 86), for any $\delta>0$ and $M>0$

$$P(\sup_{0\le t\le T}|Z_n(t)|>\delta) \le \frac{M}{\delta^2} + P(\langle Z_n, Z_n\rangle_T > M).$$

With the same $\delta = M$ and the same M as above, $P(\sup_{0\le t\le T}|Z_n(t)|>\delta)\le (3/2)\varepsilon$, from which we conclude that

$$\sup_{0\le t\le T}|Z_n(t)| = O_p\big((K\Delta_n)^{1/2}\big).$$

APPENDIX F: PROOF OF THEOREM 3.2 OF THE MAIN TEXT

For each $n\ge 1$, we write

$$\alpha_t^{(n)} = \alpha_0^{(n)} + \int_0^t \zeta_s^{(n)}\,\mathrm{d}s + \bar{\alpha}_t^{(n)} \quad \text{and} \quad \beta_t^{(n)} = \beta_0^{(n)} + \int_0^t \eta_s^{(n)}\,\mathrm{d}s + \bar{\beta}_t^{(n)},$$

where $\bar{\alpha}_t^{(n)}$ and $\bar{\beta}_t^{(n)}$ are the martingale parts of $\alpha_t^{(n)}$ and $\beta_t^{(n)}$, respectively, and the processes $\bar{\alpha}_t^{(n)}, \bar{\beta}_t^{(n)}, \zeta_t^{(n)}$ and $\eta_t^{(n)}$ satisfy Condition A.2. In particular, we we assume there are processes, say $a_{n,t}$, $b_{n,t}$, and $c_{n,t}$, such that $\mathrm{d}\langle\bar{\alpha}^{(n)}, \bar{\alpha}^{(n)}\rangle_t = a_{n,t}^2\,\mathrm{d}t$, $\mathrm{d}\langle\bar{\beta}^{(n)}, \bar{\beta}^{(n)}\rangle_t = b_{n,t}^2\,\mathrm{d}t$, and also $\mathrm{d}\langle\bar{\alpha}^{(n)}, \bar{\beta}^{(n)}\rangle_t = c_{n,t}\,\mathrm{d}t$. Moreover, we assume that that $a_{n,t}$, $b_{n,t}$, $c_{n,t}$, $\zeta_t^{(n)}$, and $\eta_t^{(n)}$ are locally bounded uniformly in n. Meaning that for all $t \le \tau_{n,m}$, where $(\tau_{n,m})_{n\ge1,m\ge1}$ is the localising sequence, we have $a_{n,t}^2 \le a_+^2$, $b_{n,t}^2 \le b_+^2$, $|c_{n,t}| \le c_+$, $|\zeta_t^{(n)}| \le \zeta_+$, and $|\eta_t^{(n)}| \le \eta_+$. As already mentioned, these bounds depend on m, but we drop this dependence from the notation. Finally, we assume that $a_{n,t}^2$, $b_{n,t}^2$, and $c_{n,t}$ are locally continuous in mean square. See Definition A.3 for continuity in mean square.

We also write $\alpha_t^{(l,n)} = \int_0^t f_{s-}^{(l,n)}\,\mathrm{d}\alpha_t^{(n)} = \bar{\alpha}_t^{(l,n)} + A_t^{(l,n)}$, with $\bar{\alpha}_t^{(l,n)} = \int_0^t f_{s-}^{(l,n)}\,\mathrm{d}\bar{\alpha}_t^{(n)}$ and $A_t^{(l,n)} = \int_0^t f_{s-}^{(l,n)}\zeta_s^{(n)}\,\mathrm{d}s$. Similarly, $\beta_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)}\,\mathrm{d}\beta_t^{(n)} = \bar{\beta}_t^{(l,n)} + B_t^{(l,n)}$ with $\bar{\beta}_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)}\,\mathrm{d}\bar{\beta}_s^{(n)}$ and $B_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)}\eta_s^{(n)}\,\mathrm{d}s$. For $l = 1, \ldots, 2K$, define

$$Z_{n,l}(t) = \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (\alpha_{t_{i+K}}^{(l,n)} - \alpha_{t_{i-K}}^{(l,n)})(\beta_{t_{i+K}}^{(l,n)} - \beta_{t_{i-K}}^{(l,n)}) + (\alpha_t^{(l,n)} - \alpha_{t_{*,l}}^{(l,n)})(\beta_t^{(l,n)} - \beta_{t_{*,l}}^{(l,n)}) - [\alpha^{(l,n)}, \beta^{(l,n)}]_t.$$

Note that since $A_t^{(l,n)}$ and $B_t^{(l,n)}$ are continuous processes of locally finite variation, $[\alpha^{(l,n)}, \beta^{(l,n)}] = [\bar{\alpha}^{(l,n)} + A^{(l,n)}, \bar{\beta}^{(l,n)} + B^{(l,n)}] = [\bar{\alpha}^{(l,n)}, \bar{\beta}^{(l,n)}]$ (Jacod and Shiryaev, 2003, Prop. I.4.49, p. 52). Therefore,

$$Z_{n,l}(t) = Z_{n,l}^{\mathrm{mg}}(t) + Z_{n,l}^{\mathrm{ds}}(t) + \breve{Z}_{n,l}^{(1)}(t) + \breve{Z}_{n,l}^{(2)}(t),$$

where,

$$\begin{aligned}
Z_{n,l}^{\mathrm{mg}}(t) &= \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (\bar{\alpha}_{t_{i+K}}^{(l,n)} - \bar{\alpha}_{t_{i-K}}^{(l,n)})(\bar{\beta}_{t_{i+K}}^{(l,n)} - \bar{\beta}_{t_{i-K}}^{(l,n)}) \\
&\qquad + (\bar{\alpha}_t^{(l,n)} - \bar{\alpha}_{t_{*,l}}^{(l,n)})(\bar{\beta}_t^{(l,n)} - \bar{\beta}_{t_{*,l}}^{(l,n)}) - [\bar{\alpha}^{(l,n)}, \bar{\beta}^{(l,n)}]_t; \\
Z_{n,l}^{\mathrm{ds}}(t) &= \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (A_{t_{i+K}}^{(l,n)} - A_{t_{i-K}}^{(l,n)})(B_{t_{i+K}}^{(l,n)} - B_{t_{i-K}}^{(l,n)}) \\
&\qquad + (A_t^{(l,n)} - A_{t_{*,l}}^{(l,n)})(B_t^{(l,n)} - B_{t_{*,l}}^{(l,n)}); \\
\breve{Z}_{n,l}^{(1)}(t) &= \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (\bar{\alpha}_{t_{i+K}}^{(l,n)} - \bar{\alpha}_{t_{i-K}}^{(l,n)})(B_{t_{i+K}}^{(l,n)} - B_{t_{i-K}}^{(l,n)}) \\
&\qquad + (\bar{\alpha}_t^{(l,n)} - \bar{\alpha}_{t_{*,l}}^{(l,n)})(B_t^{(l,n)} - B_{t_{*,l}}^{(l,n)}); \\
\breve{Z}_{n,l}^{(2)}(t) &= \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (A_{t_{i+K}}^{(l,n)} - A_{t_{i-K}}^{(l,n)})(\bar{\beta}_{t_{i+K}}^{(l,n)} - \bar{\beta}_{t_{i-K}}^{(l,n)}) \\
&\qquad + (A_t^{(l,n)} - A_{t_{*,l}}^{(l,n)})(\bar{\beta}_t^{(l,n)} - \bar{\beta}_{t_{*,l}}^{(l,n)}),
\end{aligned} \tag{F.1}$$

and we note that

$$\begin{aligned}
Z_{n,l}^{\mathrm{mg}}(t) &= \sum_{t_{i+K}\le t,\, i\equiv l[2K]} \Big\{ \int_{t_{i-K}}^{t_{i+K}} (\bar{\alpha}_{t_{i+K}}^{(l,n)} - \bar{\alpha}_{t_{i-K}}^{(l,n)})\,\mathrm{d}\bar{\beta}_s + \int_{t_{i-K}}^{t_{i+K}} (\bar{\beta}_{t_{i+K}}^{(l,n)} - \bar{\beta}_{t_{i-K}}^{(l,n)})\,\mathrm{d}\bar{\alpha}_s \Big\} \\
&\qquad + \int_{t_{*,l}}^{t} (\bar{\alpha}_t^{(l,n)} - \bar{\alpha}_{t_{*,l}}^{(l,n)})\,\mathrm{d}\bar{\beta}_s + \int_{t_{*,l}}^{t} (\bar{\beta}_t^{(l,n)} - \bar{\beta}_{t_{*,l}}^{(l,n)})\,\mathrm{d}\bar{\alpha}_s.
\end{aligned} \tag{F.2}$$

Define $Z_n(t) = (2K)^{-1}\sum_{l=1}^{2K} Z_{n,l}(t)$, and $Z_n^{\rm mg}(t) = (2K)^{-1}\sum_{l=1}^{2K} Z_{n,l}^{\rm mg}(t)$, $Z_n^{\rm ds}(t) = (2K)^{-1}\sum_{l=1}^{2K} Z_{n,l}^{\rm ds}(t)$, $\breve{Z}_n^{(j)}(t) = (2K)^{-1}\sum_{l=1}^{2K} \breve{Z}_{n,l}^{(j)}(t)$, for $j = 1, 2$. Then

$$Z_n(t) = \frac{1}{2K}\sum_{l=1}^{2K} Z_{n,l}(t) = Z_n^{\rm mg}(t) + Z_n^{\rm ds}(t) + \breve{Z}_n^{(1)}(t) + \breve{Z}_n^{(2)}(t). \tag{F.3}$$

This means that $Z_n(t)$ is the semimartingale of Eq. (3.3) of the main text. We now show that $Z_n^{\rm ds}(t)$, $\breve{Z}_n^{(1)}(t)$, and $\breve{Z}_n^{(2)}(t)$ are all $o_p((K\Delta_n)^{-1/2})$ uniformly in t as $K\Delta_n \to 0$, and that the martingale $(K\Delta_n)^{-1/2} Z_n^{\rm mg}$ satisfies condition (i), (ii), and (iii) of Theorem B.1. The claim of the theorem then follows Lemma A.2 of Appendix A.

We start with $Z_n^{\rm ds}$ and it suffices to look at one of the summands $Z_{n,l}^{\rm ds}$. Integration by parts

$$\begin{aligned}
Z_{n,l}^{\rm ds}(t) = \sum_{t_{i+K}\le t}\{&\int_{t_{i-K}}^{t_{i+K}} (A_s^{(l,n)} - A_{t_{i-K}}^{(l,n)}) g_{s-}^{(l,n)} \eta_s^{(n)}\,{\rm d}s + \int_{t_{i-K}}^{t_{i+K}} (B_s^{(l,n)} - B_{t_{i-K}}^{(l,n)}) f_{s-}^{(l,n)} \zeta_s^{(n)}\,{\rm d}s\} \\
&+ \int_{t_{*,l}}^{t} (A_s^{(l,n)} - A_{t_{*,l}}^{(l,n)}) g_{s-}^{(l,n)} \eta_s^{(n)}\,{\rm d}s + \int_{t_{*,l}}^{t} (B_s^{(l,n)} - B_{t_{*,l}}^{(l,n)}) f_{s-}^{(l,n)} \zeta_s^{(n)}\,{\rm d}s.
\end{aligned}$$

Look at one of the terms in the summand. For $t_{i+K} \le \tau_{n,m}$,

$$\begin{aligned}
{\rm E}\,|\int_{t_{i-K}}^{t_{i+K}} (A_s^{(l,n)} - A_{t_{i-K}}^{(l,n)}) g_{s-}^{(l,n)} \eta_s^{(n)}\,{\rm d}s| &\le {\rm E}\int_{t_{i-K}}^{t_{i+K}} |\int_{t_{i-K}}^{s} f_{u-}^{(l,n)} \zeta_u^{(n)}\,{\rm d}u||\eta_s^{(n)}|\,{\rm d}s \\
&\le \zeta_+\eta_+ \int_{t_{i-K}}^{t_{i+K}} (s - t_{i-K})\,{\rm d}s = \frac{\zeta_+\eta_+}{2}(t_{i+K} - t_{i-K})^2.
\end{aligned}$$

This shows that ${\rm E}\,|Z_n^{\rm ds}(t)| \le R_n^{\rm ds}(t)$ for all $t \le \tau_{n,m}$, where $R_n^{\rm ds}(t)$ is a nonnegative deterministic function such that $\sup_{t\le T} R_n^{\rm ds}(t) = O(K\Delta_n)$ as $K\Delta_n \to 0$. Since $R_n^{\rm ds}(t)$ is deterministic, it is predictable, and so Lenglart's inequality combined with a localisation argument (see Lemma A.1) yield $\sup_{t\le T}|Z_n^{\rm ds}(t)| = o_p((K\Delta_n)^{1/2})$ as $K\Delta_n \to 0$.

– Next, we turn to $\breve{Z}_n^{(1)}(t)$ and $\breve{Z}_n^{(2)}(t)$. It suffices to look at $\breve{Z}_n^{(1)}(t)$. Define $G_t^{(l,n)} = \int_0^t g_{s-}^{(l,n)}\,{\rm d}s$, and notice that $|G_t^{(l,n)} - G_s^{(l,n)}| \le |t - s|$ for all t and s. Integration by parts

$$ {\rm d}\{(\eta_s^{(n)} - \eta_{t_{i-K}}^{(n)})(G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\} = -(\eta_s^{(n)} - \eta_{t_{i-K}}^{(n)}) g_{s-}^{(l,n)}\,{\rm d}s + (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,{\rm d}\eta_s^{(n)},$$

which makes sense because $G_t^{(l,n)}$ is deterministic, so that

$$\int_{t_{i-K}}^{t_{i+K}} (\eta_s^{(n)} - \eta_{t_{i-K}}^{(n)}) g_s^{(l,n)}\,{\rm d}s = \int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,{\rm d}\eta_s^{(n)}.$$

We may then express the increments $B_{t_{i+K}}^{(l,n)} - B_{t_{i-K}}^{(l,n)}$ appearing in $\breve{Z}_{n,l}^{(1)}(t)$ (see (F.1)) as

$$\begin{aligned}
B_{t_{i+K}}^{(l,n)} - B_{t_{i-K}}^{(l,n)} &= \int_{t_{i-K}}^{t_{i+K}} g_{s-}^{(l,n)} \eta_s^{(n)}\,{\rm d}s \\
&= \eta_{t_{i-K}}^{(n)} \int_{t_{i-K}}^{t_{i+K}} g_{s-}^{(l,n)}\,{\rm d}s + \int_{t_{i-K}}^{t_{i+K}} g_{s-}^{(l,n)} (\eta_s^{(n)} - \eta_{t_{i-K}}^{(n)})\,{\rm d}s \\
&= \eta_{t_{i-K}}^{(n)} \int_{t_{i-K}}^{t_{i+K}} g_{s-}^{(l,n)}\,{\rm d}s + \int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,{\rm d}\eta_s^{(n)}.
\end{aligned}$$

We now insert this expression for $B^{(l,n)}_{t_{i+K}} - B^{(l,n)}_{t_{i-K}}$ in $\breve{Z}^{(1)}_{n,l}(t)$ and write

$$\breve{Z}^{(1)}_{n,l}(t) = c^{\mathrm{mg}}_{n,l}(t) + \breve{c}_{n,l}(t),$$

where

$$c^{\mathrm{mg}}_{n,l}(t) = \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (\bar{\alpha}^{(l,n)}_{t_{i+K}} - \bar{\alpha}^{(l,n)}_{t_{i-K}})\eta^{(n)}_{t_{i-K}} \int_{t_{i-K}}^{t_{i+K}} g^{(l,n)}_{s-}\,\mathrm{d}s$$
$$+ (\bar{\alpha}^{(l,n)}_{t} - \bar{\alpha}^{(l,n)}_{t_{*,l}})\eta^{(n)}_{t_{*,l}} \int_{t_{*,l}}^{t} g^{(l,n)}_{s-}\,\mathrm{d}s,$$

and

$$\breve{c}_{n,l}(t) = \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (\bar{\alpha}^{(l,n)}_{t_{i+K}} - \bar{\alpha}^{(l,n)}_{t_{i-K}}) \int_{t_{i-K}}^{t_{i+K}} (G^{(l,n)}_{t_{i+K}} - G^{(l,n)}_{s})\,\mathrm{d}\eta^{(n)}_s$$
$$+ (\bar{\alpha}^{(l,n)}_{t} - \bar{\alpha}^{(l,n)}_{t_{*,l}}) \int_{t_{*,l}}^{t} (G^{(l,n)}_{t} - G^{(l,n)}_{s})\,\mathrm{d}\eta^{(n)}_s,$$

so that $\breve{Z}^{(1)}_{n}(t) = (2K)^{-1}\sum_{l=1}^{2K} c^{\mathrm{mg}}_{n,l}(t) + (2K)^{-1}\sum_{l=1}^{2K}\breve{c}_{n,l}(t) = c^{\mathrm{mg}}_{n}(t) + \breve{c}_n(t)$, by which we define c^{mg}_n and $\breve{c}_n$. By the Kunita–Watanabe inequality $\langle c^{\mathrm{mg}}_{n,l_1}, c^{\mathrm{mg}}_{n,l_2}\rangle_t$ is bounded by $(\langle c^{\mathrm{mg}}_{n,l_1}, c^{\mathrm{mg}}_{n,l_1}\rangle_t \langle c^{\mathrm{mg}}_{n,l_2}, c^{\mathrm{mg}}_{n,l_2}\rangle_t)^{1/2}$, so it suffices to look at $\langle c^{\mathrm{mg}}_{n,l}, c^{\mathrm{mg}}_{n,l}\rangle_t$. For $t \le \tau_{n,m}$, it is

$$\langle c^{\mathrm{mg}}_{n,l}, c^{\mathrm{mg}}_{n,l}\rangle_t = \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (\eta^{(n)}_{t_{i-K}})^2 \Big(\int_{t_{i-K}}^{t_{i+K}} g^{(l,n)}_{s-}\,\mathrm{d}s\Big)^2 \int_{t_{i-K}}^{t_{i+K}} (f^{(l,n)}_{s-})^2\,\mathrm{d}\langle \bar{\alpha}^{(n)}, \bar{\alpha}^{(n)}\rangle_s$$
$$+ (\eta^{(n)}_{t_{*,l}})^2 \Big(\int_{t_{*,l}}^{t} g^{(l,n)}_{s-}\,\mathrm{d}s\Big)^2 \int_{t_{*,l}}^{t} (f^{(l,n)}_{s-})^2\,\mathrm{d}\langle \bar{\alpha}^{(n)}, \bar{\alpha}^{(n)}\rangle_s$$
$$\le a_+^2\eta_+^2 \sum_{t_{i+K}\le t,\, i\equiv l[2K]} (t_{i+K} - t_{i-K})^3 + a_+^2\eta_+^2 (t_{i+K} - t_{i-K})^3.$$

This shows that for $t \le \tau_{n,m}$, $\langle c^{\mathrm{mg}}_{n,l}, c^{\mathrm{mg}}_{n,l}\rangle_t = O_p((K\Delta_n)^2)$. Thus, $\langle c^{\mathrm{mg}}_{n,l_1}, c^{\mathrm{mg}}_{n,l_2}\rangle_t = O_p((K\Delta_n)^2)$ by the Kunita–Watanabe inequality, and for $t \le \tau_n$

$$\frac{1}{K\Delta_n}\langle c^{\mathrm{mg}}_n, c^{\mathrm{mg}}_n\rangle_t = \frac{1}{4K^2(K\Delta_n)} \sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K} \langle c^{\mathrm{mg}}_{n,l_1}, c^{\mathrm{mg}}_{n,l_2}\rangle_t = O_p(K\Delta_n).$$

Lenglart's inequality combined with a localisation argument (see Lemma A.1) then yield that

$$\sup_{t\le T} |c^{\mathrm{mg}}_n(t)| = o_p((K\Delta_n)^{1/2}),$$

as $K\Delta_n \to 0$. We now look at $\breve{c}_n$. The expectation of its absolute value is

$$\mathrm{E}\,|\breve{c}_{n,l}(t)| \le \sum_{t_{i+K}\le t} \mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f^{(l,n)}_{s-}\,\mathrm{d}\bar{\alpha}^{(n)}_s\Big|\,\Big|\int_{t_{i-K}}^{t_{i+K}} (G^{(l,n)}_{t_{i+K}} - G^{(l,n)}_s)\,\mathrm{d}\eta^{(n)}_s\Big|$$
$$+ \mathrm{E}\,\Big|\int_{t_{*,l}}^{t} f^{(l,n)}_{s-}\,\mathrm{d}\bar{\alpha}^{(n)}_s\Big|\,\Big|\int_{t_{*,l}}^{t} (G^{(l,n)}_{t} - G^{(l,n)}_s)\,\mathrm{d}\eta^{(n)}_s\Big|.$$

Consider one summand at the time. We now use that $\eta^{(n)}$ satisfies Condition A.2, and write $\mathrm{d}\eta^{(n)}_t = \varphi^{(n)}_t\,\mathrm{d}t + \mathrm{d}\bar{\eta}^{(n)}_t$, with $\bar{\eta}^{(n)}_t$ the martingale part of $\eta^{(n)}_t$, and let $v_+ = v_{m,+}$ and $\varphi_+ =$

$\varphi_{m,+}$ be such that $\mathrm{d}\langle\bar\eta^{(n)},\bar\eta^{(n)}\rangle_t/\mathrm{d}t \le v_+^2$ and $|\varphi_t^{(n)}| \le \varphi_+$ for all $t \le \tau_{n,m}$. By the triangle inequality

$$\begin{aligned}\mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|\,\Big|\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,\mathrm{d}\eta_s^{(n)}\Big| \\ \le \mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|\,\Big|\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,\mathrm{d}\bar\eta_s^{(n)}\Big| \\ + \mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|\,\Big|\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\varphi_s^{(n)}\,\mathrm{d}s\Big|,\end{aligned}$$

and so for $t \le \tau_{n,m}$,

$$\begin{aligned}&\mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|\,\Big|\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,\mathrm{d}\bar\eta_s^{(n)}\Big| \\ &\le \Big(\mathrm{E}\,\Big(\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big)^2\,\mathrm{E}\,\Big(\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\,\mathrm{d}\bar\eta_s^{(n)}\Big)^2\Big)^{1/2} \\ &= \Big(\mathrm{E}\,\int_{t_{i-K}}^{t_{i+K}} (f_{s-}^{(l,n)})a_{n,s}^2\,\mathrm{d}s\,\mathrm{E}\,\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})^2\,\mathrm{d}\langle\bar\eta^{(n)},\bar\eta^{(n)}\rangle_s\Big)^{1/2} \\ &\le v_+\Big(\mathrm{E}\,\int_{t_{i-K}}^{t_{i+K}} (f_{s-}^{(l,n)})a_{n,s}^2\,\mathrm{d}s\,\int_{t_{i-K}}^{t_{i+K}} (t_{i+K}-s)^2\,\mathrm{d}s\Big)^{1/2} \\ &\le a_+v_+\{(t_{i+K}-t_{i-K})(t_{i+K}-t_{i-K})^3/3\}^{1/2} = \frac{a_+v_+}{3^{1/2}}\,(t_{i+K}-t_{i-K})^2,\end{aligned}$$

and, similarly

$$\begin{aligned}&\mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|\,\Big|\int_{t_{i-K}}^{t_{i+K}} (G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)})\varphi_s^{(n)}\,\mathrm{d}s\Big| \\ &\le \varphi_+\mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|\int_{t_{i-K}}^{t_{i+K}} |G_{t_{i+K}}^{(l,n)} - G_s^{(l,n)}|\mathrm{d}s \\ &\le \frac{\varphi_+}{2}\mathrm{E}\,\Big|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big|(t_{i+K}-t_{i-K})^2 \le \frac{\varphi_+}{2}\Big\|\int_{t_{i-K}}^{t_{i+K}} f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big\|_2(t_{i+K}-t_{i-K})^2 \\ &\le \frac{\varphi_+a_+}{2}(t_{i+K}-t_{i-K})^{1/2}(t_{i+K}-t_{i-K})^2 = \frac{\varphi_+a_+}{2}(t_{i+K}-t_{i-K})^{5/2}.\end{aligned}$$

This shows that $\breve c_n(t)$ is L-dominated by a deterministic, hence predictable, process, say $\breve R_n(t)$, and that this process is such that $\breve R_n(t)/(K\Delta_n)^{1/2} = O((K\Delta_n)^{1/2})$ as $K\Delta_n \to 0$ for each t. Again, Lenglart's inequality combined with a localisation argument (as in Lemma A.1) gives that $\sup_{t\le T}|\breve c_n(t)| = o_p((K\Delta_n)^{1/2})$. In conclusion, $\breve Z_n^{(1)}(t)$ and $\breve Z_n^{(2)}(t)$ are both $o_p((K\Delta_n)^{1/2})$ uniformly in t as $K\Delta_n \to 0$, and so the sequence in (F.3) is

$$Z_n(t) = Z_n^{\mathrm{mg}}(t) + o_p((K\Delta_n)^{1/2}), \tag{F.4}$$

uniformly in t as $K\Delta_n \to 0$. We now turn to the martingale part of (F.4), namely Z_n^{mg}, and show that this sequence satisfies Condtions (i)–(iii) of Theorem B.1. Then we appeal to Lemma A.2 of Appendix A.

First we show that the predictable quadratic variation of $(K\Delta_n)^{-1/2}Z_n^{\mathrm{mg}}$ converges in probability as $K\Delta_n \to 0$. The predictable quadratic variation of Z_n^{mg} is

$$\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_t = \frac{1}{4K^2}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t.$$

Here

$$\begin{aligned}
\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t &= \int_0^t (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{*,l_1}}^{(l_1,n)})(\bar\alpha_s^{(l_2,n)} - \bar\alpha_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle \bar\beta^{c,(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\quad + \int_0^t (\bar\beta_s^{(l_1,n)} - \bar\beta_{t_{*,l_1}}^{(l_1,n)})(\bar\beta_s^{(l_2,n)} - \bar\beta_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle \bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}\rangle_s \\
&\quad + \int_0^t (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{*,l_1}}^{(l_1,n)})(\bar\beta_s^{(l_2,n)} - \bar\beta_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}\rangle_s \\
&\quad + \int_0^t (\bar\alpha_s^{(l_2,n)} - \bar\alpha_{t_{*,l_2}}^{(l_2,n)})(\bar\beta_s^{(l_1,n)} - \bar\beta_{t_{*,l_1}}^{(l_1,n)})\,\mathrm{d}\langle \bar\beta^{(l_2,n)}, \bar\alpha^{(l_1,n)}\rangle_s \\
&= \langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t^{(1)} + \langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t^{(2)} + \langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t^{(3)} + \langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t^{(4)},
\end{aligned}$$

by which we define $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_t^{(j)}$ for $j = 1, 2, 3, 4$. Start by concentrating on $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(1)}$, and the same results apply automatically to $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(2)}$. It is given by

$$\begin{aligned}
\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(1)} &= \int_0^T (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{*,l_1}}^{(l_1,n)})(\bar\alpha_s^{(l_2,n)} - \bar\alpha_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&= \int_0^T (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{*,l_1}}^{(l_1,n)})g_s^{(l_1,n)}(\bar\alpha_s^{(l_2,n)} - \bar\alpha_{t_{*,l_2}}^{(l_2,n)})g_s^{(l_2,n)}\,\mathrm{d}\langle \bar\beta^{(n)}, \bar\beta^{(n)}\rangle_s.
\end{aligned} \tag{F.5}$$

Write

$$\{t_{i-K}, t_{i+K} : i \equiv l[2K],\ K \le i \le B-K\} = \{t_{0,l}, t_{1,l}, t_{2,l}, \dots\},$$

where the indices on the right hand side are such that $t_{i,l} < t_{i+1,l}$, and let $\mathcal{G}^{(l)}$ be the set of these time points, i.e., $\mathcal{G}^{(l)} = \{t_{0,l} < t_{1,l} < t_{2,l} < \cdots\}$. With this notation we have, e.g., that

$$\sum_{K\le i\le B-K,\, i\equiv l[2K]} (\alpha_{t_{i+K}}^{(l,n)} - \alpha_{t_{i-K}}^{(l,n)})(\beta_{t_{i+K}}^{(l,n)} - \beta_{t_{i-K}}^{(l,n)}) = \sum_{t_{i+1,l}\le T} (\alpha_{t_{i+1,l}}^{(l,n)} - \alpha_{t_{i,l}}^{(l,n)})(\beta_{t_{i+1,l}}^{(l,n)} - \beta_{t_{i,l}}^{(l,n)}).$$

The time $t_{*,l}$ defined in (E.1) is now simply $t_{*,l} = \max\{t_i \in \mathcal{G}^{(l)} : t_i \le s\} = \max\{t_{i,l} : t_{i,l} \le s\}$. Attach the number $t_{-1,l} = 0$ to $\mathcal{G}^{(l)}$ if it is not already there, and suppose, without loss of generality, that $t_{i,l_1} < t_{i,l_2}$ for all i, and that $t_{0,l_1} = 0$. We can then write

$$\begin{aligned}
\langle Z^{\mathrm{mg}}_{n,l_1}, Z^{\mathrm{mg}}_{n,l_2}\rangle^{(1)}_T &= \int_0^T (\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{*,l_1}})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{*,l_2}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&= \sum_{t_{i+1,l_1}\le T} \int_{t_{i,l_1}}^{t_{i+1,l_1}} (\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{i,l_1}})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{*,l_2}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&= \sum_{i:\,t_{i+1,l_1}\le T} \Big\{ \int_{t_{i,l_1}}^{t_{i,l_2}} (\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{i,l_1}})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{i-1,l_2}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\quad + \int_{t_{i,l_2}}^{t_{i+1,l_1}} (\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{i,l_1}})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{i,l_2}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \Big\} \\
&\quad + \int_{t_{*,l_1}(T)}^{T} (\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{*,l_1}(s)})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{*,l_2}(s)})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s.
\end{aligned} \tag{F.6}$$

We now want to show that (F.6) is

$$\begin{aligned}
\langle Z^{\mathrm{mg}}_{n,l_1}, Z^{\mathrm{mg}}_{n,l_2}\rangle^{(1)}_T &= \sum_{t_{i+1,l_1}\le T} \Big\{ \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} \mathrm{d}[\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_u\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\quad + \int_{t_{i,l_2}}^{t_{i+1,l_1}} \int_{t_{i,l_2}}^{s} \mathrm{d}[\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_u\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \Big\} \\
&\quad + \int_{t_{*,l_1}(T)}^{T} \int_{(t_{*,l_1}\vee t_{*,l_2})(T)}^{s} \mathrm{d}[\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_u\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s + o_p(K\Delta_n).
\end{aligned} \tag{F.7}$$

The key is to show equalities of the type

$$\begin{aligned}
&\int_{t_{i,l_1}}^{t_{i,l_2}} (\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{i,l_1}})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{i-1,l_2}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&= \int_{t_{i,l_1}}^{t_{i,l_2}} ([\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_s - [\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_{t_{i,l_1}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s + \text{negligible},
\end{aligned} \tag{F.8}$$

and that the negligible terms are $o_p((K\Delta_n)^2)$. Recall that $t_{i-1,l_2} < t_{i,l_1}$, thus by Itô's formula

$$\begin{aligned}
&(\bar\alpha^{(l_1,n)}_s - \bar\alpha^{(l_1,n)}_{t_{i,l_1}})(\bar\alpha^{(l_2,n)}_s - \bar\alpha^{(l_2,n)}_{t_{i-1,l_2}}) - ([\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_s - [\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_{t_{i,l_1}}) \\
&\qquad = \int_{t_{i,l_1}}^{s} (\bar\alpha^{(l_1,n)}_u - \bar\alpha^{(l_1,n)}_{t_{i,l_1}})\,\mathrm{d}\bar\alpha^{(l_2,n)}_u + \int_{t_{i,l_1}}^{s} (\bar\alpha^{(l_2,n)}_u - \bar\alpha^{(l_2,n)}_{t_{i-1,l_2}})\,\mathrm{d}\bar\alpha^{(l_1,n)}_u.
\end{aligned}$$

This means that the expression in (F.8) is

$$\begin{aligned}
&\int_{t_{i,l_1}}^{t_{i,l_2}} (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})(\bar\alpha_s^{(l_2,n)} - \bar\alpha_{t_{i-1,l_2}}^{(l_2,n)})\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\quad = \int_{t_{i,l_1}}^{t_{i,l_2}} ([\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_s - [\bar\alpha^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_{t_{i,l_1}})\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\qquad + \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\qquad + \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_2,n)} - \bar\alpha_{t_{i-1,l_2}}^{(l_2,n)})\,\mathrm{d}\bar\alpha_u^{(l_1,n)}\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s.
\end{aligned} \tag{F.9}$$

We now consider the two last terms on the right hand side of this expression, and look at

$$\begin{aligned}
&\int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}\,\mathrm{d}\langle \bar\beta^{(l_1,n)}, \bar\beta^{(l_2,n)}\rangle_s \\
&\quad = \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} b_{n,s}^2\,\mathrm{d}s \\
&\quad = \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} b_{n,t_{i,l_1}}^2\,\mathrm{d}s \\
&\qquad + \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} (b_{n,s}^2 - b_{n,t_{i,l_1}}^2)\,\mathrm{d}s.
\end{aligned} \tag{F.10}$$

Consider the two terms on the right in (F.10) separately, starting with the first term. Define the functions

$$G_t^{(l_1,l_2)} = \int_0^t g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)}\,\mathrm{d}s, \qquad l_1, l_2 = 1, \dots 2K,$$

and note that since $|g_s^{(l_1,n)}| \le 1$ for all s, the functions $G_t^{(l_1,l_2)}$ are Lipschitz with constant 1, that is,

$$|G_t^{(l_1,l_2)} - G_s^{(l_1,l_2)}| \le |t-s|, \quad \text{for all } t, s.$$

Integration by parts yields

$$\begin{aligned}
&\mathrm{d}\{\int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})\} \\
&\quad = -\int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)}\,\mathrm{d}s \\
&\qquad + (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})(\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_s^{(l_2,n)}.
\end{aligned}$$

Integrating from t_{i,l_1} to t_{i,l_2},

$$\begin{aligned}
&\int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)}\,\mathrm{d}s \\
&\quad = \int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})(\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_s^{(l_2,n)}.
\end{aligned}$$

Since $G_s^{(l_1,l_2)}$ is deterministic it is predictable, so the right hand side of this expression is a martingale. Then, for $t_{i,l_2} < \tau_{n,m}$, we find the following bound for the right hand side of (F.10),

$$\begin{aligned}
&\|\int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar{\alpha}_u^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar{\alpha}_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} b_{n,t_{i,l_1}}^2\,\mathrm{d}s\|_2^2 \\
&\qquad \le b_+^2 \|\int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar{\alpha}_s^{(l_2,n)}\|_2^2 \\
&\qquad = b_+^2\,\mathrm{E}\int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})^2(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})^2\,\mathrm{d}\langle\bar{\alpha}^{(l_2,n)},\bar{\alpha}^{(l_2,n)}\rangle_s \\
&\qquad = b_+^2\,\mathrm{E}\int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})^2(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})^2 (f_s^{(l,n)})^2 a_{n,s}^2\,\mathrm{d}s \\
&\qquad \le a_+^2 b_+^2\,\mathrm{E}\int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})^2(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})^2\,\mathrm{d}s \\
&\qquad \le a_+^2 b_+^2\,\mathrm{E}\int_{t_{i,l_1}}^{t_{i,l_2}} (t_{i,l_2} - s)^2\,\mathrm{E}\,(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})^2\,\mathrm{d}s \\
&\qquad \le a_+^2 b_+^2 \int_{t_{i,l_1}}^{t_{i,l_2}} (t_{i,l_2} - s)^2\,\mathrm{E}\,(\langle\bar{\alpha}^{(l_1,n)},\bar{\alpha}^{(l_1,n)}\rangle_s - \langle\bar{\alpha}^{(l_1,n)},\bar{\alpha}^{(l_1,n)}\rangle_{t_{i,l_1}})\,\mathrm{d}s \\
&\qquad \le a_+^4 b_+^2 \int_{t_{i,l_1}}^{t_{i,l_2}} (t_{i,l_2} - s)^2 (s - t_{i,l_1})\,\mathrm{d}s = \frac{a_+^4 b_+^2}{12}(t_{i,l_2} - t_{i,l_1})^4.
\end{aligned} \tag{F.11}$$

Since the martingale increments are uncorrelated, this gives

$$\begin{aligned}
&\mathrm{E}\,\Big(\sum_{i:\,t_{i+1,l_1}\le\tau_{n,m}} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar{\alpha}_u^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar{\alpha}_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} b_{n,s}^2\,\mathrm{d}s\Big)^2 \\
&\qquad \le b_+^2\,\mathrm{E}\,\Big(\sum_{i:\,t_{i+1,l_1}\le\tau_{n,m}} \int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar{\alpha}_s^{(l_2,n)}\Big)^2 \\
&\qquad = b_+^2 \sum_{i:\,t_{i+1,l_1}\le\tau_{n,m}} \mathrm{E}\,\Big(\int_{t_{i,l_1}}^{t_{i,l_2}} (G_{t_{i,l_2}}^{(l_1,l_2)} - G_s^{(l_1,l_2)})(\bar{\alpha}_s^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar{\alpha}_s^{(l_2,n)}\Big)^2 \\
&\qquad \le \frac{a_+^4 b_+^2}{12} \sum_{i:\,t_{i+1}\le\tau_{n,m}} (t_{i,l_2} - t_{i,l_1})^4.
\end{aligned}$$

By Chebyshev's inequality we have that for any $\varepsilon > 0$,

$$\begin{aligned}
&P\{|\sum_{i:\,t_{i+1,l_1}\le T} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar{\alpha}_u^{(l_1,n)} - \bar{\alpha}_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar{\alpha}_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} b_{n,s}^2\,\mathrm{d}s| \ge \varepsilon\} \\
&\qquad\qquad \le \frac{1}{\varepsilon^2}\frac{a_+^4 b_+^2}{12} \sum_{i:\,t_{i+1}\le\tau_n} (t_{i,l_2} - t_{i,l_1})^4 + P(\tau_{n,m} \ne T),
\end{aligned}$$

which shows that the first term on the right in (F.10) is

$$\sum_{i:\,t_{i+1,l_1}\le T}\int_{t_{i,l_1}}^{t_{i,l_2}}\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}g_{s-}^{(l_1,n)}g_{s-}^{(l_2,n)}b_{n,s}^2\,\mathrm{d}s=o_p(K\Delta_n),$$

as $K\Delta_n\to 0$. We now turn to the second term on the right in (F.10). For $t_{i,l_1}\le s<\tau_{n,m}$,

$$\begin{aligned}
&\|\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}g_{s-}^{(l_1,n)}g_{s-}^{(l_2,n)}(b_{n,s}^2-b_{n,t_{i,l_1}}^2)\|_1\\
&\quad\le\|\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}\|_2\,\|g_{s-}^{(l_1,n)}g_{s-}^{(l_2,n)}(b_{n,s}^2-b_{n,t_{i,l_1}}^2)\|_2\\
&\quad=\|\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}\|_2\,|g_{s-}^{(l_1,n)}g_{s-}^{(l_2,n)}|\,\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2\\
&\quad\le\|\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}\|_2\,\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2\\
&\quad=\big(\mathrm{E}\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})^2(f_u^{(l_2,n)})^2a_{n,u}^2\,\mathrm{d}u\big)^{1/2}\,\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2 \qquad \text{(F.12)}\\
&\quad\le a_+\big(\mathrm{E}\int_{t_{i,l_1}}^{s}(\langle\bar\alpha^{(l_1,n)},\bar\alpha^{(l_1,n)}\rangle_u-\langle\bar\alpha^{(l_1,n)},\bar\alpha^{(l_1,n)}\rangle_{t_{i,l_1}})\,\mathrm{d}u\big)^{1/2}\,\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2\\
&\quad\le a_+^2\big(\int_{t_{i,l_1}}^{s}(u-t_{i,l_1})\,\mathrm{d}u\big)^{1/2}\,\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2\\
&\quad=\frac{a_+^2}{\sqrt2}(s-t_{i,l_1})\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2\le\frac{a_+^2}{\sqrt2}(s-t_{i,l_1})\sup_{t_{i,l_1}\le s\le t_{i,l_2}}\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2.
\end{aligned}$$

From this we get that for $t_{i,l_2}<\tau_{n,m}$,

$$\begin{aligned}
&\|\int_{t_{i,l_1}}^{t_{i,l_2}}\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}g_{s-}^{(l_1,n)}g_{s-}^{(l_2,n)}(b_{n,s}^2-b_{n,t_{i,l_1}}^2)\,\mathrm{d}s\|_1\\
&\qquad\le\int_{t_{i,l_1}}^{t_{i,l_2}}\|\int_{t_{i,l_1}}^{s}(\bar\alpha_u^{(l_1,n)}-\bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)}g_{s-}^{(l_1,n)}g_{s-}^{(l_2,n)}(b_{n,s}^2-b_{n,t_{i,l_1}}^2)\,\mathrm{d}s\|_1\\
&\qquad\le\frac{a_+^2}{\sqrt2}\int_{t_{i,l_1}}^{t_{i,l_2}}(s-t_{i,l_1})\,\mathrm{d}s\sup_{t_{i,l_1}\le s\le t_{i,l_2}}\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2\\
&\qquad=\frac{a_+^2}{2^{3/2}}(t_{i,l_2}-t_{i,l_1})^2\sup_{t_{i,l_1}\le s\le t_{i,l_2}}\|b_{n,s}^2-b_{n,t_{i,l_1}}^2\|_2,
\end{aligned}$$

from which

$$
\begin{aligned}
&\| \sum_{i:t_{i+1,l_1}\le\tau_{n,m}} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} (b_{n,s}^2 - b_{n,t_{i,l_1}}^2)\,\mathrm{d}s\|_1 \\
&\qquad \le \sum_{i:t_{i+1,l_1}\le\tau_{n,m}} \int_{t_{i,l_1}}^{t_{i,l_2}} \| \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} (b_{n,s}^2 - b_{n,t_{i,l_1}}^2)\,\mathrm{d}s\|_1 \\
&\qquad \le \frac{a_+^2}{2^{3/2}} \sum_{i:t_{i+1,l_1}\le\tau_{n,m}} (t_{i,l_2} - t_{i,l_1})^2 \sup_{t_{i,l_1}\le s\le t_{i,l_2}} \|b_{n,s}^2 - b_{n,t_{i,l_1}}^2\|_2 \\
&\qquad \le \frac{a_+^2}{2^{3/2}} \Big(\sum_{i:t_{i+1,l_1}\le\tau_{n,m}} (t_{i,l_2} - t_{i,l_1})^2\Big) \sup_{0\le|t-s|\le K\Delta_n} \|b_{n,t}^2 - b_{n,s}^2\|_2.
\end{aligned}
$$

By Markov's inequality

$$
\begin{aligned}
&P\Big\{\Big| \sum_{i:t_{i+1,l_1}\le T} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\alpha_u^{(l_1,n)} - \alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} (b_{n,s}^2 - b_{n,t_{i,l_1}}^2)\,\mathrm{d}s\Big| \ge \varepsilon\Big\} \\
&\qquad \le \frac{1}{\varepsilon}\frac{a_+^2}{2^{3/2}} \Big(\sum_{i:t_{i+1,l_1}\le\tau_n} (t_{i,l_2} - t_{i,l_1})^2\Big) \sup_{0\le|t-s|\le K\Delta_n} \|b_{n,t}^2 - b_{n,s}^2\|_2 + P(\tau_{n,m}\ne T).
\end{aligned}
$$

By mean square continuity of $b_{n,s}^2$, we get

$$
\sum_{t_{i+1,l_1}\le T} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\alpha_u^{(l_1,n)} - \alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\alpha_u^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} (b_{n,s}^2 - b_{n,t_{i,l_1}}^2)\,\mathrm{d}s = o_p(K\Delta_n).
$$

This completes the proof of (F.7), and, obviously, the same holds for $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(2)}$.

We must now show that similar results apply to $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(3)}$ and $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(4)}$. It suffices to look at $\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(3)}$. In analogy with (F.6), we can write

$$
\begin{aligned}
\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(3)} &= \int_0^T (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{*,l_1}}^{(l_1,n)})(\bar\beta_s^{(l_2,n)} - \bar\beta_{t_{*,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}\rangle_s \\
&= \sum_{i:t_{i+1,l_1}\le T} \Big\{ \int_{t_{i,l_1}}^{t_{i,l_2}} (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})(\bar\beta_s^{(l_2,n)} - \bar\beta_{t_{i-1,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}\rangle_s \\
&\qquad + \int_{t_{i,l_2}}^{t_{i+1,l_1}} (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})(\bar\beta_s^{(l_2,n)} - \bar\beta_{t_{i,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}\rangle_s \Big\} \\
&\qquad\quad + \int_{t_{*,l_1}(T)}^{T} (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{*,l_1}(s)}^{(l_1,n)})(\bar\beta^{(l_2,n)} - \bar\beta_{t_{*,l_2}(s)}^{(l_2,n)})\,\mathrm{d}\langle\bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}\rangle_s.
\end{aligned}
$$

And, in analogy with (F.9) and (F.10), we write

$$\int_{t_{i,l_1}}^{t_{i,l_2}} (\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})(\bar\beta_s^{(l_2,n)} - \bar\beta_{t_{i-1,l_2}}^{(l_2,n)})\,\mathrm{d}\langle\bar\beta^{(l_1,n)},\bar\alpha^{(l_2,n)}\rangle_s$$
$$= \int_{t_{i,l_1}}^{t_{i,l_2}} ([\bar\alpha^{(l_1,n)},\bar\beta^{(l_2,n)}]_s - [\bar\alpha^{(l_1,n)},\bar\beta^{(l_2,n)}]_{t_{i,l_1}})\,\mathrm{d}\langle\bar\beta^{(l_1,n)},\bar\alpha^{(l_2,n)}\rangle_s + r_{n,i,1} + r_{n,i,2},$$

where the remainder terms are

$$r_{n,i,1} = \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\beta_u^{(l_2,n)}\,\mathrm{d}\langle\bar\beta^{(l_1,n)},\bar\alpha^{(l_2,n)}\rangle_s$$
$$= \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\beta_u^{(l_2,n)} g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} c_{n,t_{i,l_1}}\,\mathrm{d}s$$
$$+ \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\beta_u^{(l_2,n)} g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} (c_{n,s} - c_{n,t_{i,l_1}})\,\mathrm{d}s;$$
$$r_{n,i,2} = \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\beta_u^{(l_2,n)} - \bar\beta_{t_{i-1,l_2}}^{(l_2,n)})\,\mathrm{d}\bar\alpha_u^{(l_1,n)}\,\mathrm{d}\langle\bar\beta^{(l_1,n)},\bar\alpha^{(l_2,n)}\rangle_s.$$

The remainder terms $r_{n,i,1}$ and $r_{n,i,2}$ can be dealt with in the same manner, so we concentrate on $r_{n,i,1}$ in the following. Define the functions $H_t^{(l_1,l_2)} = \int_0^t g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)}\,\mathrm{d}s$, and note that $|H_t^{(l_1,l_2)} - H_s^{(l_1,l_2)}| \le |t-s|$ for all t, s. Then, for $t_{i,l_2} < \tau_{n,m}$,

$$\int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\beta_u^{(l_2,n)} g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} c_{n,t_{i,l_1}}\,\mathrm{d}s$$
$$= \int_{t_{i,l_1}}^{t_{i,l_2}} (H_{t_{i,l_2}}^{(l_1,l_2)} - H_s^{(l_1,l_2)})(\bar\alpha_s^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\beta_s^{(l_2,n)}.$$

A derivation similar to that of (F.11), gives that

$$\mathrm{E}\Big(\sum_{i:\,t_{i+1,l_1} \le \tau_{n,m}} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\mathrm{d}\bar\beta_u^{(l_2,n)} g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} c_{n,t_{i,l_1}}\,\mathrm{d}s\Big)^2$$
$$\le \frac{a_+^2 b_+^2 |c_+|}{12} \sum_{i:\,t_{i+1,l_1} \le \tau_{n,m}} (t_{i,l_2} - t_{i,l_1})^4.$$

Looking back at the derivation in (F.12), we now also see that

$$\Big\| \sum_{i:\,t_{i+1,l_1} \le \tau_{n,m}} \int_{t_{i,l_1}}^{t_{i,l_2}} \int_{t_{i,l_1}}^{s} (\bar\alpha_u^{(l_1,n)} - \bar\alpha_{t_{i,l_1}}^{(l_1,n)})\,\mathrm{d}\bar\beta_u^{(l_2,n)} g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} (c_{n,s} - c_{n,t_{i,l_1}})\,\mathrm{d}s \Big\|_1$$
$$\le \frac{a_+ b_+}{2^{3/2}} \Big(\sum_{i:\,t_{i+1,l_1} \le \tau_{n,m}} (t_{i,l_2} - t_{i,l_1})^2 \Big) \sup_{0 \le |t-s| \le K\Delta_n} \|c_{n,t} - c_{n,s}\|_2.$$

By the same localisation techniques used previously, this establishes that

$$\begin{aligned}
&\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(3)} + \langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T^{(4)} \\
&= \int_0^T ([\bar\alpha^{(l_1,n)}, \bar\beta^{(l_2,n)}]_s - [\bar\alpha^{(l_1,n)}, \bar\beta^{(l_2,n)}]_{t_{*,l_1}\vee t_{*,l_2}}) g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} \,\mathrm{d}\langle \bar\beta^{(n)}, \bar\alpha^{(n)}\rangle_s \\
&+ \int_0^T ([\bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_s - [\bar\beta^{(l_1,n)}, \bar\alpha^{(l_2,n)}]_{t_{*,l_1}\vee t_{*,l_2}}) f_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} \,\mathrm{d}\langle \bar\alpha^{(n)}, \bar\beta^{(n)}\rangle_s + o_p(K\Delta_n).
\end{aligned}$$

Setting all of the above together, we have shown that

$$\begin{aligned}
\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_T &= \frac{1}{4K^2}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\langle Z_{n,l_1}^{\mathrm{mg}}, Z_{n,l_2}^{\mathrm{mg}}\rangle_T \\
&= \frac{1}{4K^2}\sum_{l_1=1}^{2K}\sum_{l_2=1}^{2K}\int_0^T\int_{t_{*,l_1}\vee t_{*,l_2}}^s \Bigg\{ f_{u-}^{(l_1,n)} f_{u-}^{(l_2,n)} g_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)} \,\mathrm{d}[\bar\alpha^{(n)}, \bar\alpha^{(n)}]_u \frac{\mathrm{d}\langle\bar\beta^{(n)}, \bar\beta^{(n)}\rangle_s}{\mathrm{d}s} \\
&\qquad + g_{u-}^{(l_1,n)} g_{u-}^{(l_2,n)} f_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)} \,\mathrm{d}[\bar\beta^{(n)}, \bar\beta^{(n)}]_u \frac{\mathrm{d}\langle\bar\alpha^{(n)}, \bar\alpha^{(n)}\rangle_s}{\mathrm{d}s} \\
&\qquad + (f_{u-}^{(l_1,n)} g_{u-}^{(l_2,n)})(g_{s-}^{(l_1,n)} f_{s-}^{(l_2,n)}) \,\mathrm{d}[\bar\alpha^{(n)}, \bar\beta^{(n)}]_u \frac{\mathrm{d}\langle\bar\beta^{(n)}, \bar\alpha^{(n)}\rangle_s}{\mathrm{d}s} \\
&\qquad + (f_{u-}^{(l_2,n)} g_{u-}^{(l_1,n)})(f_{s-}^{(l_1,n)} g_{s-}^{(l_2,n)}) \,\mathrm{d}[\bar\beta^{(n)}, \bar\alpha^{(n)}]_u \frac{\mathrm{d}\langle\bar\alpha^{(n)}, \bar\beta^{(n)}\rangle_s}{\mathrm{d}s} \Bigg\} \mathrm{d}s + o_p(K\Delta_n) \\
&= (K\Delta_n)\int_0^T \kappa_s^{(n)}\,\mathrm{d}s + o_p(K\Delta_n),
\end{aligned}$$

using that, by Condition A.2, the predictable quadratic variations $\langle\bar\alpha^{(n)}, \bar\alpha^{(n)}\rangle_s$, $\langle\bar\beta^{(n)}, \bar\beta^{(n)}\rangle_s$, and $\langle\bar\alpha^{(n)}, \bar\beta^{(n)}\rangle_s$ are absolutely continuous. By assumption, there is an $\mathcal{F}$-measurable process κ_s such that $\int_0^t \kappa_s^{(n)}\,\mathrm{d}s \to_p \int_0^t \kappa_s\,\mathrm{d}s$ for all t when $K\Delta_n \to 0$ and $K\to\infty$ as n tends to infinity. This shows that $(K\Delta_n)^{-1/2} Z_n(t)^{\mathrm{mg}}$ satisfies condition (i) of Theorem B.1.

We now turn to condition (ii) of Theorem B.1, that is the Lindeberg condition. To verify that this condition holds, we appeal to condition (ii)$''$ of Corollary B.2. We must verify that the sequence $(K\Delta_n)^{-1/2} Z_n^{\mathrm{mg}}$ is P-UT, that $\sup_{0\le t\le T}|\Delta Z_n^{\mathrm{mg}}(t)| = o_p(K\Delta_n)$ as $n\to\infty$, and that $\sup_{0\le t\le T}\mathrm{E}\,(K\Delta_n)^{-1}(\Delta Z_n^{\mathrm{mg}}(t))^2 < \infty$ for all n. We have seen that $(K\Delta_n)^{-1}\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_t$ converges in probability, hence also in distribution, to the continuous and increasing process $\int_0^t \kappa_s\,\mathrm{d}s$. By Jacod and Shiryaev (2003, Theorem VI.3.37, p. 354) this yields process convergence of $(K\Delta_n)^{-1}\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle$ to $\int_0^\cdot \kappa_s\,\mathrm{d}s$, which means that $(K\Delta_n)^{-1}\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle = O_p(1)$ in the sense of Definition A.1. To see that $(K\Delta_n)^{-1/2} Z_n^{\mathrm{mg}}$ is P-UT, let H^n be any predictable process with $|H_t^n| \le 1$, and let $H^n \cdot Z_n^{\mathrm{mg}}(t)$ be the elementary stochastic integral (see Jacod and Shiryaev (2003, p. 377) for both definitions). Now, $\mathrm{E}\,(H^n\cdot Z_n^{\mathrm{mg}}(t))^2 = \mathrm{E}\,(H^n\cdot Z_n^{\mathrm{mg}}(t))^2 = \mathrm{E}\,(H^n)^2\cdot[Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}]_t \le \mathrm{E}\,\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_t$, so by Lenglart's inequality, for any $\varepsilon, \eta > 0$,

$$\begin{aligned}
P((K\Delta_n)^{-1/2}|H^n\cdot Z_n^{\mathrm{mg}}(t)| \ge \varepsilon) &\le P((K\Delta_n)^{-1}\sup_{t\le T}|H^n\cdot Z_n^{\mathrm{mg}}(t)|^2 \ge \varepsilon^2) \\
&\le \eta/\varepsilon^2 + P((K\Delta_n)^{-1}\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_T \ge \eta),
\end{aligned}$$

and that $(K\Delta_n)^{-1/2} Z_n^{\mathrm{mg}}$ is P-UT follows from the definition (Jacod and Shiryaev, 2003, Definition VI.6.1, p. 377), because $(K\Delta_n)^{-1}\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_T$ is tight.

The jumps of $Z_n^{\mathrm{mg}}(t)$ are

$$\Delta Z_n^{\mathrm{mg}}(t) = \frac{1}{2K}\sum_{l=1}^{2K} \Delta Z_{n,l}^{\mathrm{mg}}(t).$$

Using (F.2) we see that

$$\begin{aligned}
\Delta Z_{n,l}^{\mathrm{mg}}(t) &= (\bar\alpha_t^{(l,n)} - \bar\alpha_{t_{*,l}}^{(l,n)})\Delta\bar\beta_t^{(l,n)} + (\bar\beta_t^{(l,n)} - \bar\beta_{t_{*,l}}^{(l,n)})\Delta\bar\alpha_t^{(l,n)}\\
&= (\bar\alpha_t^{(l,n)} - \bar\alpha_{t_{*,l}}^{(l,n)}) g_{t-}^{(l,n)}\Delta\bar\beta_t^{(n)} + (\bar\beta_t^{(l,n)} - \bar\beta_{t_{*,l}}^{(l,n)}) f_{t-}^{(l,n)}\Delta\bar\alpha_t^{(n)}\\
&= \Big(\int_{t_{*,l}}^t f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big)\, g_{t-}^{(l,n)}\Delta\bar\beta_t^{(n)} + \Big(\int_{t_{*,l}}^t g_{s-}^{(l,n)}\,\mathrm{d}\bar\beta_s^{(n)}\Big)\, f_{t-}^{(l,n)}\Delta\bar\alpha_t^{(n)}.
\end{aligned}$$

For $t \le \tau_{n,m}$, we have that by the Itô isometry,

$$\mathrm{E}\Big(\int_{t_{*,l}}^t f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\Big)^2 \le \mathrm{E}\int_{t_{*,l}}^t (f_{s-}^{(l,n)})^2\,\mathrm{d}\langle\bar\alpha^{(n)},\bar\alpha^{(n)}\rangle_s \le a_+^2(t-t_{*,l}) \le a_+^2 K\Delta_n,$$

which shows that $\big(\int_{t_{*,l}}^t f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}\big)^2$ is locally L-dominated (Jacod and Shiryaev, 2003, Definition I.3.29, p. 35) by the predictable process $\int_{t_{*,l}}^t (f_{s-}^{(l,n)})^2\,\mathrm{d}\langle\bar\alpha^{(n)},\bar\alpha^{(n)}\rangle_s$, and that the latter is $O_p((K\Delta_n)^{1/2})$. Therefore, by Condition A.2,

$$P(\sup_{t\le T}|\int_{t_{*,l}}^t f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}| \ge \varepsilon) \le P(\sup_{t\le \tau_{n,m}}|\int_{t_{*,l}}^t f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}| \ge \varepsilon) + P(\tau_{n,m} < T),$$

which shows that $\sup_{t\le T}|\int_{t_{*,l}}^t f_{s-}^{(l,n)}\,\mathrm{d}\bar\alpha_s^{(n)}|$ and $\sup_{t\le T}|\int_{t_{*,l}}^t g_{s-}^{(l,n)}\,\mathrm{d}\bar\beta_s^{(n)}|$ are $O_p((K\Delta_n)^{1/2})$ as $n\to\infty$. But for any $\varepsilon > 0$,

$$\sup_{t\le T}|\Delta\bar\beta_t^{(n)}| \le \sup_{t\le T}|\Delta\bar\beta_t^{(n)}| I\{|\Delta\bar\beta_t^{(n)}| \ge \varepsilon\} + \varepsilon \le \int_{|x|\ge\varepsilon}|x|\,\mu_\beta^n([0,T]\times\mathrm{d}x) + \varepsilon \xrightarrow{p} \varepsilon,$$

as $n\to\infty$ by the Lindeberg condition in Eq. (3.5) of the main text, combined with Lenglart's inequality. But since $\varepsilon > 0$ was arbitrary, $\sup_{t\le T}|\Delta\bar\beta_t^{(n)}| = o_p(1)$ as $n\to\infty$, and we conclude that $\sup_{t\le T}|\Delta Z_{n,l}^{\mathrm{mg}}(t)| = o_p((K\Delta_n)^{1/2})$. For the last condition, by Jacod and Shiryaev (2003, Theorem I.4.47(c), p. 52), the triangle inequality, and using that $[Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}]_t^2$ is an increasing process, we have that for any t,

$$\mathrm{E}\,(\Delta Z_n^{\mathrm{mg}}(t))^2 = \mathrm{E}\,\Delta[Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}]_t \le 2\mathrm{E}\,[Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}]_T = 2\mathrm{E}\,\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_T,$$

but from Appendix E (i.e., the proof of Theorem 3.1 of the main text), we have that $\mathrm{E}\,\langle Z_n^{\mathrm{mg}}, Z_n^{\mathrm{mg}}\rangle_T \lesssim 4TK\Delta_n$. Thus $\sup_{t\le T}(K\Delta_n)^{-1}\mathrm{E}\,(\Delta Z_n^{\mathrm{mg}}(t))^2 < \infty$ for all n, and we conclude that Condition (ii)$''$ of Corollary B.2 is satisfied, and therefore the Lindeberg condition of Theorem B.1 is also satisfied.

It remains to check Condition (iii) of Theorem B.1, namely that

$$(K\Delta_n)^{-1/2}\langle Z_n^{\mathrm{mg}}, X^n\rangle_t \xrightarrow{p} 0, \quad \text{for each } t\in[0,T],$$

where X^n is a sequence of bounded martingales. It is enough to check this condition for a sequence of processes X^n that is either a sequence of Wiener processes, or a sequence of Poisson processes (this is a consequence of a representation theorem in Cohen and Elliott (2015, Theorem 14.5.7, p. 360)). This means that the sequence X^n has predictable quadratic variation $\langle X^n, X^n\rangle_t = t$ or $\langle X^n, X^n\rangle_t = \int_0^t \lambda_s \,\mathrm{d}s$ for some deterministic and nonnegative function λ. For simplicity of the exposition we assume that X^n is a sequence of Wiener processes. By the Kunita–Watanabe inequality (Protter, 2004, Theorem II.25, p. 69), for $h > 0$ and $t + h \leq \tau_{n,m}$,

$$
\begin{aligned}
|\langle \bar\beta^{(l,n)}, X^n\rangle_{t+h} - \langle \bar\beta^{(l,n)}, X^n\rangle_t| &\leq \Big(\int_t^{t+h} (g_{s-}^{(l,n)})^2 b_{n,s}^2 \,\mathrm{d}s \,\langle X^n, X^n\rangle_{(t,t+h]}\Big)^{1/2} \\
&\leq \Big(\int_t^{t+h} b_{n,s}^2 \,\mathrm{d}s \, h\Big)^{1/2} \leq b_+ h,
\end{aligned}
$$

writing $\langle X^n, X^n\rangle_{(t,t+h]} = \langle X^n, X^n\rangle_{t+h} - \langle X^n, X^n\rangle_t$. Thus, $\mathrm{d}|\langle \bar\beta^{(l,n)}, X^n\rangle_t|/\mathrm{d}t \leq b_+$ for all $t \leq \tau_{n,m}$, where

$$
|\langle \bar\beta^{(l,n)}, X^n\rangle_t| = \langle \bar\beta^{(l,n)}, X^n\rangle_t^+ + \langle \bar\beta^{(l,n)}, X^n\rangle_t^-,
$$

with $\langle \bar\beta^{(l,n)}, X^n\rangle_t^+$ and $\langle \bar\beta^{(l,n)}, X^n\rangle_t^-$ the positive and the negative part of $\langle \bar\beta^{(l,n)}, X^n\rangle_t$, respectively. For a fixed l and $t_{i+K} \leq \tau_n$

$$
\begin{aligned}
\Big\| \int_{t_{i-K}}^{t_{i+K}} (\bar\alpha_s^{(l,n)} - \bar\alpha_{t_{i-K}}^{(l,n)}) \,\mathrm{d}\langle \bar\beta^{(l,n)}, X^n\rangle_s \Big\|_1 &\leq \mathrm{E} \int_{t_{i-K}}^{t_{i+K}} |(\bar\alpha_s^{(l,n)} - \bar\alpha_{t_{i-K}}^{(l,n)})| \,\mathrm{d}|\langle \bar\beta^{(l,n)}, X^n\rangle_s| \\
&\leq b_+ \int_{t_{i-K}}^{t_{i+K}} \|(\bar\alpha_s^{(l,n)} - \bar\alpha_{t_{i-K}}^{(l,n)})\|_1 \,\mathrm{d}s \\
&\leq b_+ \int_{t_{i-K}}^{t_{i+K}} \|(\bar\alpha_s^{(l,n)} - \bar\alpha_{t_{i-K}}^{(l,n)})\|_2 \,\mathrm{d}s \\
&\leq a_+ b_+ \int_{t_{i-K}}^{t_{i+K}} (s - t_{i-K}) \,\mathrm{d}s \\
&= \frac{a_+ b_+}{2} (t_{i+K} - t_{i-K})^2,
\end{aligned}
$$

where for the third inequality we have used Hölder’s inequality. Then, for $t < \tau_{n,m}$,

$$
\begin{aligned}
\|\langle Z_{n,l}^{\mathrm{mg}}, X^n\rangle_t\|_1 \leq \sum_{t_{i+K} \leq t,\, i \equiv l[2K]} & \Big\{ \Big\| \int_{t_{i-K}}^{t_{i+K}} (\bar\alpha_s^{(l,n)} - \bar\alpha_{t_{i-K}}^{(l,n)}) \,\mathrm{d}\langle \bar\beta^{(l,n)}, X^n\rangle_s \Big\|_1 \\
&+ \Big\| \int_{t_{i-K}}^{t_{i+K}} (\bar\beta_s^{(l,n)} - \bar\beta_{t_{i-K}}^{(l,n)}) \,\mathrm{d}\langle \bar\alpha^{(l,n)}, X^n\rangle_s \Big\|_1 \Big\} \\
&+ \Big\| \int_{t_{*,l}}^{t} (\bar\alpha_s^{(l,n)} - \bar\alpha_{t_{*,l}}^{(l,n)}(h)) \,\mathrm{d}\langle \bar\beta^{(l,n)}, X^n\rangle_s \Big\|_1 \\
&+ \Big\| \int_{t_{*,l}}^{t} (\bar\beta_s^{(l,n)} - \bar\beta_{t_{*,l}}^{(l,n)}) \,\mathrm{d}[\bar\alpha^{(l,n)}, X^n]_s \Big\|_1 \\
\leq a_+ b_+ \sum_{t_{i+K} \leq t,\, i \equiv l[2K]} & (t_{i+K} - t_{i-K})^2 + a_+ b_+ (t - t_{*,l})^2 = O(K\Delta_n),
\end{aligned}
$$

hence $\langle Z_n^{\mathrm{mg}}, X^n\rangle_t = (2K)^{-1} \sum_{l=1}^{2K} \langle Z_{n,l}^{\mathrm{mg}}, X^n\rangle_t = o_p((K\Delta_n)^{1/2})$ for each $t \leq \tau_{n,m}$, and the third requirement of Theorem B.1 follows from Condition A.2 and a localisation argument such as that in Lemma A.1.

We have now shown that the martingale sequence $(K\Delta_n)^{-1/2} Z_n^{\mathrm{mg}}$ converges stably in law to an $\mathcal{F}$-conditional Gaussian martingale with variance process $\int_0^t \kappa_s \, \mathrm{d}s$. And since $(K\Delta_n)^{-1/2} Z_n = (K\Delta_n)^{-1/2} Z_n^{\mathrm{mg}} + o_p(1)$ uniformly in t as $K\Delta_n \to 0$, the claim of the theorem follows from Lemma A.2.

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